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MARINE GRAVIMETRY FROM AN AIRCRAFT

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(1954)

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ABSTRACT

A method is presented for rapidly surveying the gravity field of the earth over that portion of the surface covered with water. The techniques cannot approach the accuracies attainable with stationary gravimeters. The oceans of the earth comprise the majority of the gravimetrically uncharted portion of its surface. Only gravity measurements at sea ~~will be~~ discussed in this work.

To provide rapid coverage independent of oceanic disturbances, a low-flying aircraft rather than a ship will be used to transport the gravimeter. The required accuracy will be obtained through the use of state-of-the-art equipment. The use of an aircraft presents problems in accurate determination of airspeed, altitude, latitude, longitude, and the first and second derivatives of these quantities. However, an advantage derived through the use of an aircraft is the presence of a vertical displacement reference, an altimeter which measures height above the surface of the water. This reference makes possible a method for nullifying

effects of vertical accelerations when measurements of gravity are averaged over a four to five minute period. This averaging technique will provide enough gravity field information to permit a world-wide gravity survey.

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CHAPTER 1

PURPOSE, PROPOSAL, AND GEOMETRY

Scientists have attempted to determine the size and shape of the earth for centuries. Most current mathematical representations of the shape of the earth have been in use for over a quarter of a century. The past five years have seen an increase in the data available for a more accurate determination of the shape of the earth through the analysis of artificial satellite orbits. While these data are helpful in determining a more accurate mathematical model for the earth, additional information in the form of local gravity measurements is needed. Dr. Lloyd Thompson of the Geophysics Research Directorate, Air Force Cambridge Research Center, has stated that an accuracy in the value of gravity of 3 to 4 mgal is needed over the entire surface of the earth to construct a more accurate mathematical model of the earth. The most rapid means of collecting this data is by aircraft.

The authors will investigate a method by which this information can be gathered in an aircraft with

a stabilized platform and an inertial navigation system. No attempt will be made to compete with land-based gravimeters, where an accuracy of .1 mgal, according to Dr. Thompson, can be obtained through the use of stationary gravity meters.

The solution of two problems is fundamental in this thesis. First, gravity cannot be distinguished from vertical accelerations of the gravity meter. Because of this, it is impossible to determine the value of gravity by inertial measurements alone. Some form of radiation measurement is essential to separate the desired quantity, g, from the total acceleration of the gravity meter. Second, the velocity of the gravity measuring device with respect to the earth is needed to correct the meter readings. The east-west component of the aircraft's velocity adds vectorially to the eastward tangential velocity caused by the rotation of the earth, thereby increasing or decreasing the radial acceleration. Chapter 4 will treat this problem in detail.

The authors have limited this thesis to the determination of gravity over sea for several reasons: (1) Land-based gravimeters can give accuracies an order of magnitude better than can be accomplished with moving meters. (2) Over land, the aircraft altitude above sea level cannot be determined with

sufficient accuracy. Over water, a radar device can give this information very readily. (3) Shipborne gravimeters can presently give accuracies of only ±10 mgal (4,7,16). This is not sufficiently accurate to be beneficial in improving the present model of the size and shape of the earth.

The idea of measuring gravity anomalies from an aircraft is not new (20,21,24,25,26), but a method providing consistently accurate readings has not been developed. Current projects in this area are using some of the most advanced techniques available in navigation and data processing. However, at the same time, gravimeters are being used which seem to be nothing more than unstabilized meters of the sea-type modified and shock-mounted to ride in an airplane. Many attempts (2,7,10,12,16,17,27) are being made to measure gravity on ships at sea using these unstabilized meters. In an aircraft, the problem of accurate determination of gravity is greatly increased, due to the greater velocities involved.

A suggestion made by Dr. John Hovorka of the M. I. T. Instrumentation Laboratory inspired this attempt to increase airborne gravimeter accuracy through state-of-the-art stabilized platforms. The stabilized platform could be used in either an aircraft

or a ship, but an aircraft has the inherent advantage of having a reference, i.e., the surface of the ocean, other than the medium in which it is traveling. Because of this, long-period vertical accelerations, which are present in both aircraft and ships (and are indistinguishable from gravity), can be compensated in an aircraft. These factors, along with the latest techniques in airspeed, altitude, and position determination, provide a method for reducing measurement errors to a minimum. The effect of these errors will be analyzed in Chapter 4.

The method for determining the shape of the earth from known values of gravity is attributed to Stokes (14). His theorem states that

$$N_o = \frac{1}{4\pi g r} \int_o^s F_o \, \delta g \, dS \qquad (1\text{-}1)$$

N_o = distance from the geoid to the earth spheroid, i.e., the earth ellipsoid.

g = mean value of gravity over the geoid.

r = mean value of the radius vector over the geoid.

F_o = a summation of Legendre polynomials as derived in Appendix A.

δg = gravity anomaly.

dS = element of surface on the earth's spheroid.

Three features of this theorem are of interest in this

thesis. First, in the derivation (Appendix A) of Stokes' Theorem, all masses are assumed to be inside the geoid. This is in keeping with the earth model as presented by Jeffreys (15), in which all the masses outside the geoid are considered condensed in an infinitely thin layer just inside the geoid. However, this mass displacement is so small that the difference between Jeffreys' model (15) and the model used by Heiskanen and Vening-Meinesz (14) may be neglected. In this thesis, the Heiskanen-Vening-Meinesz model will be used. Second, the method of Stokes can be used to determine only the shape of the earth. The size of the earth cannot be determined. Third, and of primary importance to this thesis, is the fact that errors increase as one goes farther from the measured values of gravity. Over ocean areas, thousands of miles from known gravity measurements, the errors become significant. Airborne gravity measurements can help by filling in the values of gravity over the sea. Then the shape of the earth can be improved by reducing the errors in the extrapolation by Stokes' Theorem.

Since this thesis deals with the measurement of gravity, it is important to define exactly what is meant by this term. Gravity is the vector sum of the earth's gravitational attraction and the centrifugal

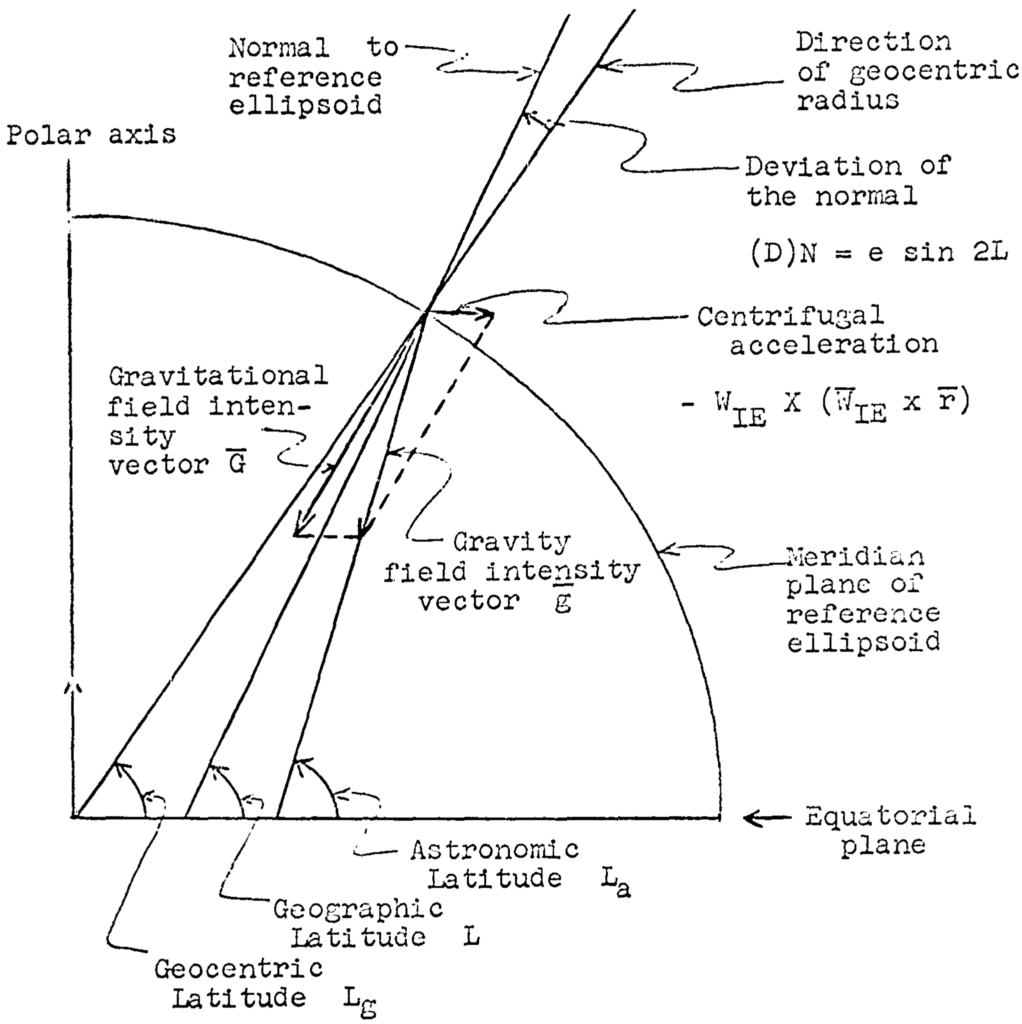

Figure 1-1

Relationship Between Gravitation and Gravity

force caused by the earth's daily rotation. The primary acceleration is the mass attraction as defined by Newton's universal law of gravitation,

$$\bar{f} = -\frac{Gm_1 m_2}{r_{12}^3} \bar{r}_{12} \tag{1-2}$$

The centrifugal force term is at most only 1/3 of one per cent of the value due to mass attraction. Because of this rotation, the earth cannot be considered a spheroid. Instead, it is flattened at the poles and bulged at the equator.

The gravity potential at any point is a scalar quantity whose maximum rate of change is the force of gravity at that point. An equipotential surface is one on which no work is done against gravity when a mass is moved between two points on it. The value of the gravity potential on such a surface is constant and the gravity force vector at any point is normal to such a surface. The mathematics of the earth's theoretical equipotential surfaces and the gravity field have been studied in many references (14,15,22). One particular equipotential surface coincides with mean sea level. This surface is called the geoid. Over land areas, the geoid is the surface the oceans would assume if narrow canals were cut through all continents. It is convenient for the purpose of this thesis that Heiskanen and Vening-Meinesz's calculations use the geoid as a reference.

Since all airborne measurements will be made with respect to sea level, the measurements are the same with respect to the geoid.

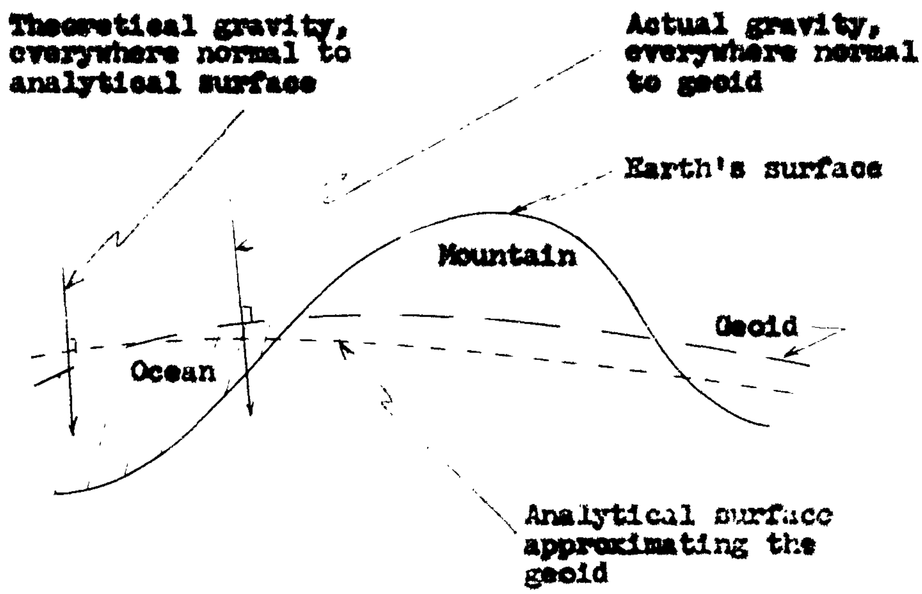

Figure 1-2

Geodetic Surfaces

Since the geoid is somewhat irregular due to gravity anomalies, it is best to work with an analytical surface that closely approximates the geoid. Many such surfaces are in use (14), all of which are in general agreement with the surface developed by Heiskanen and Vening-Meinesz. The spheroid of revolution used has the following properties:

a) It has the same flattening, $f = (a-b)/a = 1/297.00 = 0.0033670$, as the geoid.

b) It has the same mass, M, as the earth. This mass is symmetrical about the polar axis with respect to the equatorial plane.

c) The spheroid's center of gravity coincides with that of the earth.

This particular family of spheroids of revolution may be given by

$$r_p = a_p \left\{ 1 - f_p \sin^2 L_g - \left[f \left(\frac{5}{2} c' - 2f \right) + \frac{D}{a^4} \right] \times 1/4 \sin^2 2 L_g \right\} \qquad (1\text{-}3)$$

where from Figure 1-3,

Sub p = an external point

a_p = semi-major axis of analytical external equipotential surface

b_p = semi-minor axis

c' = a constant= $W_{ie}^2 \; a^3/(kM)$

D = a small constant

f_p = external flattening

$P = \overline{r}_p - \overline{r}$

This family of reference spheroids of revolution can easily be changed into a family of ellipsoids of revolution for relatively low altitudes. Since the method proposed in this thesis is restricted to low altitudes, the resulting reference ellipsoid will be used.

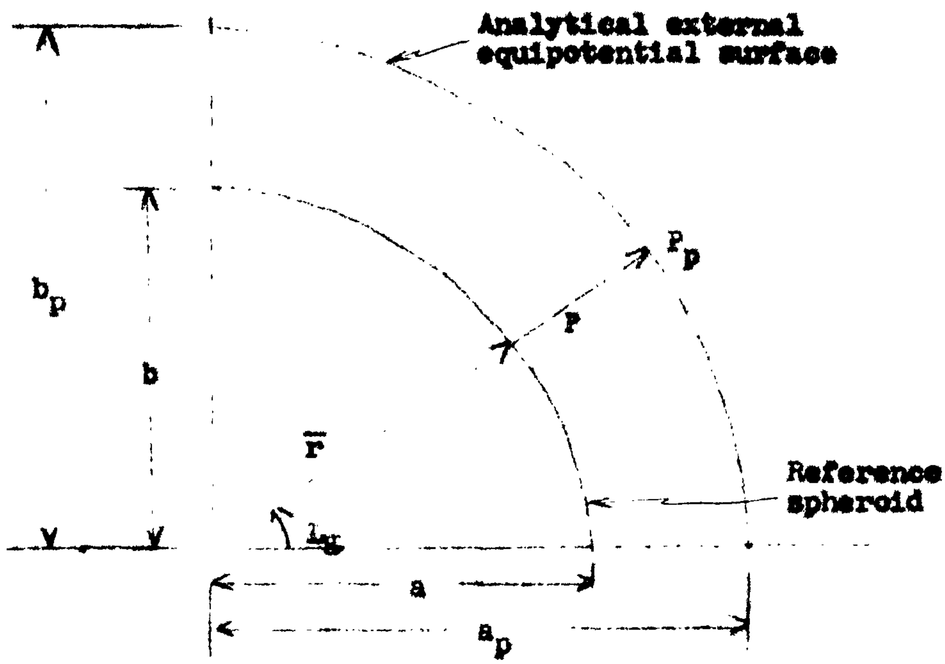

Figure 1-3

The Reference Spheroid

$$r_p = a_p \left[1 - f_p \sin^2 L_g - \frac{3}{8} f_p^2 \sin^2 2L_g \right] \qquad (1\text{-}4)$$

A value of flattening, f, must be assumed in order to compute the value of gravity on the surface of the ellipsoid. This value was chosen as 1/297 at Madrid in 1924, and is used to compute the international gravity formula. The international gravity formula, as accepted in the General Assembly of the International Union of Geodesy and Geophysics at Stockholm in 1930, is

$$g = g_e \left(1 + \beta \sin^2 L_g + \epsilon \sin^2 2L_g\right) \qquad (1\text{-}5)$$

where

g_e = value of gravity at the equator

β = coefficient of principal latitude term

ϵ = correction term

Using the values for g_e, β, and ϵ, as computed by Heiskanen, Somigliana, and Cassinis (14), the gravity formula becomes

$$g = 978.0490\,(1 + 0.0052884 \sin^2 L_g - 0.0000059 \sin^2 2L_g)\ \text{cm/sec}^2 \tag{1-6}$$

Thus far, the attraction of the sun and moon have been disregarded. Their attraction (tidal effect) has an influence on the measured value of gravity. When the sun or moon is above the point at which gravity is being measured, their attraction will cause the value of gravity to be less than the value which would be obtained if the sun and moon were beneath the point.

In Figure 1-4, the moon is closer to point P than to the center of the earth. Therefore, the attraction of the moon is greater at P than at the center of the earth. The horizontal and vertical components are

$$h_m = km_m \left[\frac{\sin z'_m}{r'^2_m} - \frac{\sin z_m}{r_m^2} \right] \tag{1-7}$$

$$V_m = km_m \left[\frac{\cos z'_m}{r'^2_m} - \frac{\cos z_m}{r^2_m} \right] \qquad (1\text{-}8)$$

where

k = Newtonian gravitational constant

m = mass of the moon

By eliminating r'_m, z'_m, and k in the above equations, we get

$$h = \frac{3}{2} km_m \frac{a}{r_m^3} \sin 2z_m = \frac{3}{2} g \frac{m_m}{M} \frac{a}{r_m^3} \times R^2 \sin 2z_m \qquad (1\text{-}9)$$

$$v = 3km \frac{a}{r^3} \left(\cos^2 z_m - \frac{1}{3}\right) = 3g \frac{m_m}{M} \frac{A}{r_m^3} R^2 \left(\cos^2 z_m - \frac{1}{3}\right) \qquad (1\text{-}10)$$

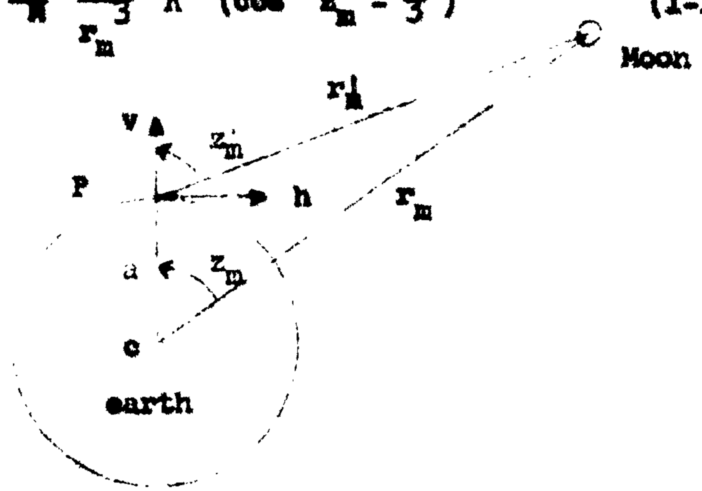

Figure 1-4

Derivation of the Tidal Effect

where

R = average earth radius

M = mass of earth

The equations for the tidal effects of the sun are similar. For values of $^{m}m/M = 1/81$, $^{m}s/M = 333,400$, $a/r_m = 1/60$, and $a/r_s = 1/23,500$, the maximum accelerations of the sun and the moon are

$$h_m = \frac{1}{11,800,000}\ g \qquad v_m = \frac{1}{8,800,000}\ g$$

$$h_s = \frac{1}{25,600,000}\ g \qquad v_s = \frac{1}{19,200,000}\ g$$

Defining a gal equal to the acceleration of one centimeter per second squared, the maximum errors due to the sun and moon are only 0.11 mgal (1 mgal $= 10^{-3}$ gal) and 0.05 mgal, respectively. Since these are well within the accuracies expected in the proposed gravity measuring device, they will be disregarded. For land-based gravimetric measurements, where accuracies of .1 mgal are expected, they cannot be disregarded without further consideration.

Since the value of gravity of interest is that at the surface of the earth, the geoid, some method must be considered for correcting gravity measurements made in an aircraft to sea level values. If we consider only the main term of gravity, g, we obtain the following value for gravity at sea level:

$$g_o = k \frac{M}{R^2} \tag{1-11}$$

The gravity g at point P, a distance h above P_o, is equal to

$$g = k \frac{M}{(R + h)^2} = \frac{kM}{R^2} \left(1 - 2 \frac{h}{R} + 3 \frac{h^2}{R^2} + \ldots\right)$$

$$g = g_o \left(1 - 2 \frac{h}{R} + 3 \frac{h^2}{R^2} - \ldots\right) \tag{1-12}$$

Thus the free air reduction $g_o - g$, or g_f

$$g_f = g_o - g = 2 \frac{g_m}{R_o} h \left(1 - \frac{3}{2} \frac{h}{R_o} + \ldots\right) \tag{1-13}$$

where g_m and R_o are the average values for gravity and earth radius. For altitudes below 6000 feet, the second term of the expression for g_f can be disregarded with errors less than 0.3 mgal, in which case,

$$g_f = 0.09406 \text{ h mgal}$$

where h is the height above the geoid in feet. For more accurate corrections, and corrections at higher altitudes,

$$g_f = 0.09406 \text{ h } \left(1 - \frac{3}{2} \frac{h}{20.9 \times 10^6} + \ldots\right)$$

$$g_f = 0.09406 \text{ h } (1 - 7.17 \times 10^{-4} \text{ h} + \ldots)$$

$$g_f = 0.09406 \text{ h} - 6.75 \times 10^{-9} \text{ h}^2 \text{ mgal} \tag{1-14}$$

At an altitude of 10,000 feet, where it is proposed in this thesis to do the gravity measurements, the

error due to omitting the h^2 term is

$$-6.75 \times 10^{-9} \times (10{,}000)^2 = -.675 \text{ mgal}$$

This error of 2/3 mgal is large enough so that the approximation used by previous authors (6,26) that the h^2 term may be disregarded is not valid for this thesis.

CHAPTER 2

A METHOD TO MEASURE GRAVITY AT SEA FROM A MOVING BASE

An aircraft flying at approximately 10,000 feet carrying a stabilized platform which is Schuler tuned will be the method used herein to transport the gravity measuring device. This gravity measuring device will be either a PIGA or PIPA with its input axis along the reference vertical as established and maintained by the stabilized platform. The Schuler tuning will practically eliminate the effects of horizontal accelerations which contribute heavily to the errors in shipborne gravimetry techniques. The only errors in this arrangement due to horizontal accelerations will be caused by the system's inability to maintain the vertical exactly.

During the measurement period, the aircraft will be flown under the most ideal conditions possible. That is, it will be operating on autopilot at a constant power setting and with no center of gravity shifting (personnel or otherwise). Over the surface of the water, turbulence due to convection will be negligible. Moments due to wind shear will be the only degrading factor, and this will be negligible under ideal weather

conditions. Any random accelerations, either vertical or horizontal, such as those due to atmospheric turbulence, will be of short duration--well under thirty seconds. These will average to zero over a four or five minute period, if the Schuler tuned vertical is undamped. Thus, the effects of random accelerations on the gravimeter are not negligible, but will average to zero over a five minute period.

Instead of using magnetic field information for a heading reference, and a barometric pressure reference for altitude hold as inputs to the automatic pilot, we will use better information which is available. This information comes from the stabilized reference in the case of heading information, and from a radar or laser altimeter in the case of altitude information. This heading reference also provides greater accuracy in position determination, thereby increasing our overall accuracy in gravity measurement. This is discussed in detail in Chapter 3.

An aircraft operating on autopilot will experience short period oscillations both laterally and longitudinally. The period of these oscillations is less than thirty seconds. These short period motions present no problems, as their effects on aircraft position and altitude average to zero over a four to six minute span of time. There is one oscillation in

pitch which has a period of approximately one minute. This so-called phugoid mode presents a problem, since the altitude change of the aircraft may vary ±5 to ±10 feet, depending on the particular aircraft and airspeed involved. Since there is a direct relationship between the error in altitude and the error in gravity measurement (10 feet corresponds to 1 mgal), some method to correct for the change in altitude and the vertical accelerations must be determined.

The simplest method to correct for vertical acceleration would be to average the reading over the measurement period. However, if the measurement period begins and ends at different phases of the oscillation, this average will be incorrect. For instance, suppose the measurement period begins and ends as shown in Figure 2-1, part A; the average will not be zero. If, on the other hand, we adjust our measuring period to integral values of the oscillation period, the average effects (acceleration and position) of the oscillation will be zero. This is shown in Figure 2-1, part B. If the average value of gravity is desired, this would be one method of nulling the effects of the changes in altitude. The method seems simple, but making it operational is more difficult. The problem of pinpointing the beginning (or any phase) of the oscillation presents the difficulty. If we know this, we can adjust our

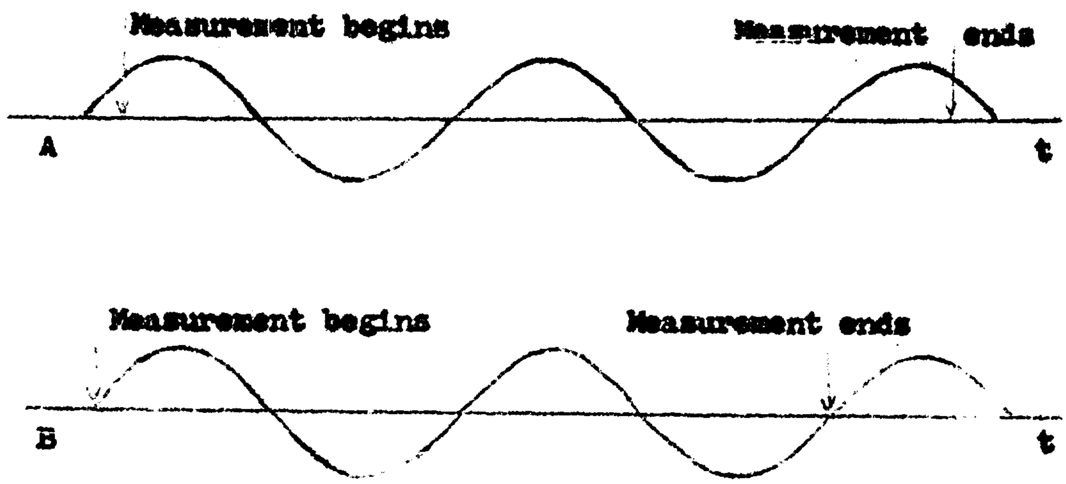

Figure 2-1

Measurement of Phugoid Period

measuring cycle to cover integral periods of the oscillation. This type of longitudinal aircraft oscillation is approximately sinusoidal. Using this fact and an accurate altimeter, the crossings of the reference altitude by the aircraft can be determined most accurately. This accuracy is limited only by the altimeter accuracy. A device called the hypsometer can determine altitude with an accuracy of ±2 feet. This device uses barometric pressure and the boiling point of toluene to attain this accuracy. A still more accurate indication of altitude can be obtained through the use of a laser. The ability of

laser to transmit a very narrow beam (approximately 1 milliradian) (11) permits it to achieve greater accuracy than radar. Altitude accuracy to within $\pm$ 1 foot* can be obtained through the use of laser.

The averaging technique over an integral number of oscillation periods can be mechanized as follows: begin the measurements as the aircraft passes through the reference altitude. End the measurement period at a predetermined, even number of crossings later. Alternately, we could begin the measurements the same as before, and end them by sensing a crossing in the same direction as the crossing at the beginning, after a time predicted by the number of periods we desire to average over. If we want to average the gravity over a specified surface length, a restriction will be placed on the number of integral cycles of pitch oscillations which we average over. Adjustment of airspeed and perhaps selection of aircraft (for both airspeed and phugoidal mode periods) can be used to overcome these restrictions.

If, instead of wanting an average over a five minute interval, we want to know the average value of gravity over a one minute period, random effects will not average to zero. Assume for the moment that our stabilizing system can maintain the vertical with no

*Accuracy estimated by Mr. Seiegienny of the Instrumentation Laboratory.

error at all times, and we can determine our altitude to within one foot. On this stabilized platform we place an accelerometer capable of measuring specific force to within 10^{-6} g (1 mgal). If we know our position within one-half mile, and the vertical acceleration within one foot per second squared at all times, we can get a measurement of gravity accurate to within two or three mgals. All these ifs and assumptions are impossible to transform into hardware. The accuracy of maintaining the vertical is not the big problem here, nor is the altimeter accuracy. The best accuracy attainable in navigation is discussed in another section of this chapter. While this aspect certainly is a degrading factor to our overall accuracy, it does not present the obstacles that the requirement for continuous vertical acceleration does.

Short period averages of gravity readings are more desirable than the five minute averages, however, the necessity for vertical acceleration information makes this technique impossible at present. A double differentiation of altitude information from a radar altimeter is extremely difficult, if not impossible. The noise cannot be separated from the surface return of this equipment to permit differentiation. Laser position information cannot be differentiated either, since it is a discontinuous process. Laser transmits

a pulse every five seconds when used in altimetry. On the basis of this consideration, this thesis will be concerned only with the averaging of measurements made of an integral number of phugoidal periods.

To define gravity, since it is a vector quantity, we must specify both magnitude and direction. If we were to measure specific force along a line maintained perpendicular to the geoid and were able to correct for all non-gravity-field forces, the result would be the magnitude of the gravity vector. If such an intensity is measured at a height above the geoid, and compared with the intensity at the same point on the reference ellipsoid, the difference is called the "free air gravity anomaly". If these free air anomalies are determined over a large area, then the "deflections of the vertical" can be computed. The deflection of the vertical is the angle between the normal to the geoid (local vertical), and the normal to the reference ellipsoid. The value of this angle averages under fifteen seconds over a continental land mass, and is assumed to be near this over the oceans. This seems to be a valid assumption since the geoid and reference ellipsoid are nearly parallel in most cases. However, there are ocean deeps where the deflection of the vertical is known to be in excess of one minute.

Since the deflection of the vertical can be com-

puted if we know the free air gravity anomaly, we will measure only the latter.

To determine the reference gravity magnitudes as a function of latitude, the international gravity formula (adopted in 1930) is used.

$$g = g_e \left(1 + \beta \sin^2 L_g - \epsilon \sin^2 2L_g\right) \qquad (2\text{-}1)$$

where

L_g = geographic latitude

g_e = intensity of gravity at the equator
= 978.049 gals

gal = 1 cm/sec^2

β = 0.0052884

ϵ = 0.0000059 at surface of reference ellipsoid

From this equation, one sees that the reference gravity field is symmetrical about the earth's polar axis. The intensity ranges from 978.049 gals at the equator to 983.054 gals at the poles, and each parallel of latitude represents a constant value of gravity intensity on the reference ellipsoid. Wilmoth (26) shows that curvature of the normals to the reference surface is negligible, particularly at the altitudes at which this system will operate. He also derives the following expression for the reference gravity field intensity as a function of height (in feet).

$$G_r = G_e \left(1 + 10^{-6}\left[5.2888 \sin^2 L_g - 5.9 \sin^2 2L_g - 0.0963\, h\right]\right) \tag{2-2}$$

where

G_r = reference gravity as a function of L_g , G_e , and h

h = altitude in feet above reference surface (geoid).

In order to find the free air gravity anomaly only, we must know L_g and h to determine reference gravity intensity and the magnitude of the earth's gravity vector at that point. From this intensity anomaly, we can by Stokes' formula determine the shape of the geoid and/or the deflection of the vertical. At this point there remains the problem of determining the actual gravity intensity with the required ±3 to 4 mgal accuracy.

One of the most critical requirements placed on a system which measures only the intensity of the gravity field vector is the maintenance of an appropriate reference. Our problem is somewhat simplified by the existence of systems which can maintain a reference vertical to within very close tolerances. Actually, errors up to five minutes in the vertical will result in erroneous gravity readings to the extent of ± 1 mgal. Present systems can maintain the vertical to at least within 25% of this figure. Actual

system accuracies will be discussed later.

Maintaining proper orientation of a reference coordinate frame in a vehicle moving above the surface of the earth is no easy proposition. First, we must look at what we're attempting to use the reference for. A reference that keeps one axis normal to the geoid is the prime requisite. This is primary because we are attempting to measure specific force along this axis. From these data, accurate determination of the free air gravity anomaly can be made. As a secondary requirement, it is desired that a reliable heading reference be maintained for an autopilot input.

These two requirements indicate that the ideal reference coordinate system in this case would be a local geographic frame. The Z axis of this system is along the normal to the reference ellipsoid and the X axis is in the meridian plane. This orientation, if maintained constant, would provide proper positioning of our force sensor, and also the desired heading reference.

While the geographic frame maintains its position constant with respect to the earth (that is, the three axes point to north, east, and down continuously), it is never stationary with respect to inertial space. This motion is due to both the rotation of the earth, and any translation of the reference frame over the surface

of the earth.

Newton's formulation of his laws of mechanics is based on inertial space. He implicitly defines inertial space as any coordinate frame in which the acceleration of a particle is proportional to the net force acting on it. For instance, if a point mass were suspended in this reference space, the force which it would exert on its support equals the vector sum of the gravitational forces, and the inertial reaction forces acting on the mass. Each of these inertial reaction forces is the negative of the product of the mass, and a particular acceleration sustained with respect to inertial space.

The gravitational forces experienced by a mass near the earth as a result of the moon and sun are less than 10^{-9} times the force due to the earth's presence. Our measuring device is not sensitive to forces of this magnitude. Hence, the gravitational pull of the sun and moon as shown in Chapter 1 can be neglected in this thesis, as can any forces exerted by any other bodies in the solar system. Thus, the accuracy required greatly simplifies the problem of measuring the intensity of the earth's gravity field. For now, if we can orient the measurement axis normal to the geoid instead of toward the center of the earth, we can find the intensity of the gravity field; providing, of course,

we can determine all the inertial reaction forces.

$$|g| = |a|_t - |a|_r \quad \text{or} \quad |f|_g = |f|_t - |f|_r$$

$$m|g| = m|a|_t - m|a|_r \qquad (2\text{-}3)$$

where

$|g|$ = magnitude of earth's gravity field

$|a|_t$ = magnitude of total acceleration

$|a|_r$ = magnitude of reaction acceleration

The measurement of gravity intensity through the use of a stabilized platform to maintain the vertical presents one outstanding problem to any acceptable scheme. This problem is caused by the use of accelerometers to maintain the required vertical indication. In other words, the gravity field is being utilized as a reference for the maintenance of a vertical along which we hope to measure the actual gravity field intensity. This seems like an impossible situation. However, since we are only interested in magnitude (and not direction) of gravity, a solution is realizable.

The extensive use of accelerometers in the problem solution requires a brief discussion of the operation principles of a linear accelerometer. Most present accelerometers contain a suspended mass with freedom of motion along one axis (input axis). The motion of such a device is determined by the sum of the gravita-

tional and inertial reaction forces acting along the input axis.

Thus the output of such a linear accelerometer is proportional to the component of total specific force along its input axis. Such devices will be used not only to determine specific forces along the three axes of the reference coordinate frame (geographic), but will also be used to generate correction signals to maintain proper orientation of the reference frame.

Performance of Linear Accelerometers

$$\left[m_e \, (\ddot{\bar{R}}_{Ee})_I\right]_{ia} = - K \, \bar{R}_{Pe} - B \, (\dot{\bar{R}}_{Pe})_{case} + m_e \, (\bar{G}_{Ee})$$

+ other terms due to the gravitational fields of the sun and moon (these terms are negligible for the purposes of this thesis) (2-4)

where

m_e = the mass of the seismic element

$(\ddot{\ })_I$ = the second time derivative with respect to inertial space

$(\dot{\ })_{ca}$ = the first time derivative with respect to a frame fixed within the instrument case

$[\ \ ]_{ia}$ = Component of the vector along the input axis of the accelerometer

K = the elastic restraint coefficient

B = the damping coefficient

$\bar{G}_{Ee}$ = the gravitational field at (e) the center

of the seismic element due to the gravitational field of the earth

From Figure 2-2

$$\bar{R}_{Ee} = \bar{R}_{EP} + \bar{R}_{Pe} \tag{2-5}$$

where

$\bar{R}_{Pe}$ = a displacement vector of point P relative to arbitrary point in the inertial reference frame

The second time derivative of Equation (2-5) relative to the inertial frame and its resolution relative to a frame fixed in the accelerometer centered at point P gives

$$\left[\ddot{\bar{R}}_{Ee}\right]_I = \left[\ddot{\bar{R}}_{EP}\right] + \left[\ddot{\bar{R}}_{Pe}\right]_{ca} + \bar{W}_{Ica} \times (\bar{W}_{Ica} \times \bar{R}_{Pe})$$

$$+ 2\bar{W}_{Ica} \times \left[\dot{\bar{R}}_{Pe}\right]_{ca} + \left[\dot{\bar{W}}_{ca}\right]_I \times \bar{R}_{Pe} \tag{2-6}$$

where

$\bar{W}_{Ica}$ = the angular velocity of the axis of the case relative to inertial space.

Substituting this equation into Equation (2-4) gives

$$\left[\bar{G}_{Ee} - (\ddot{\bar{R}}_{EP})_I\right]_{ia} = \left[\ddot{\bar{R}}_{Pe}\right]_{ca} + \frac{B}{M_e}\left[\dot{\bar{R}}_{Pe}\right]_{ca} + \frac{K}{M_e}\bar{R}_{Pe}$$

$$+ \left[\bar{W}_{Ica} \times (W_{Ica} \times \bar{R}_{Pe})\right]_{ia} \tag{2-7}$$

This equation can be further simplified through

assumptions valid for the specific scheme for usage. K can be assumed (or made) large enough so that its natural frequency is very much higher (30-50 cycles per second) than any motion of the aircraft. Therefore, instrument dynamics may be neglected. In the system under consideration, $\overline{W}_{Ica}$ will be negligible since base motion is isolated through the use of gimbals. First and second time derivatives of $\left[\overline{R}_{Pe}\right]_{ca}$ are assumed negligible in comparison with $\overline{R}_{Pe}$. The last equation therefore reduces to the following expression:

$$\left[a_{EP} - (R_{EP})_I\right]_{ia} = \frac{K}{m_e}\,\overline{R}_{Pe} \qquad (2\text{-}8)$$

$$\left(\text{Since } \overline{R}_{Pe} << \overline{R}_{EP} \text{ or } \overline{R}_{Eo}\,,\ \overline{a}_{EP} = \overline{a}_{Eo}\right)$$

Thus, the specific force along the input axis of the accelerometer is equal to a constant factor multiplied by the displacement of the element along the input axis.

If the accelerometer is to generate an output signal proportional to the specific force experienced along its input axis, the output as a voltage may be written as follows:

$$e_{out} = S_a\,(SF)_{ia} + \text{measurement uncertainties}$$

where $(SF)_{ia}$ = specific force along the input axis

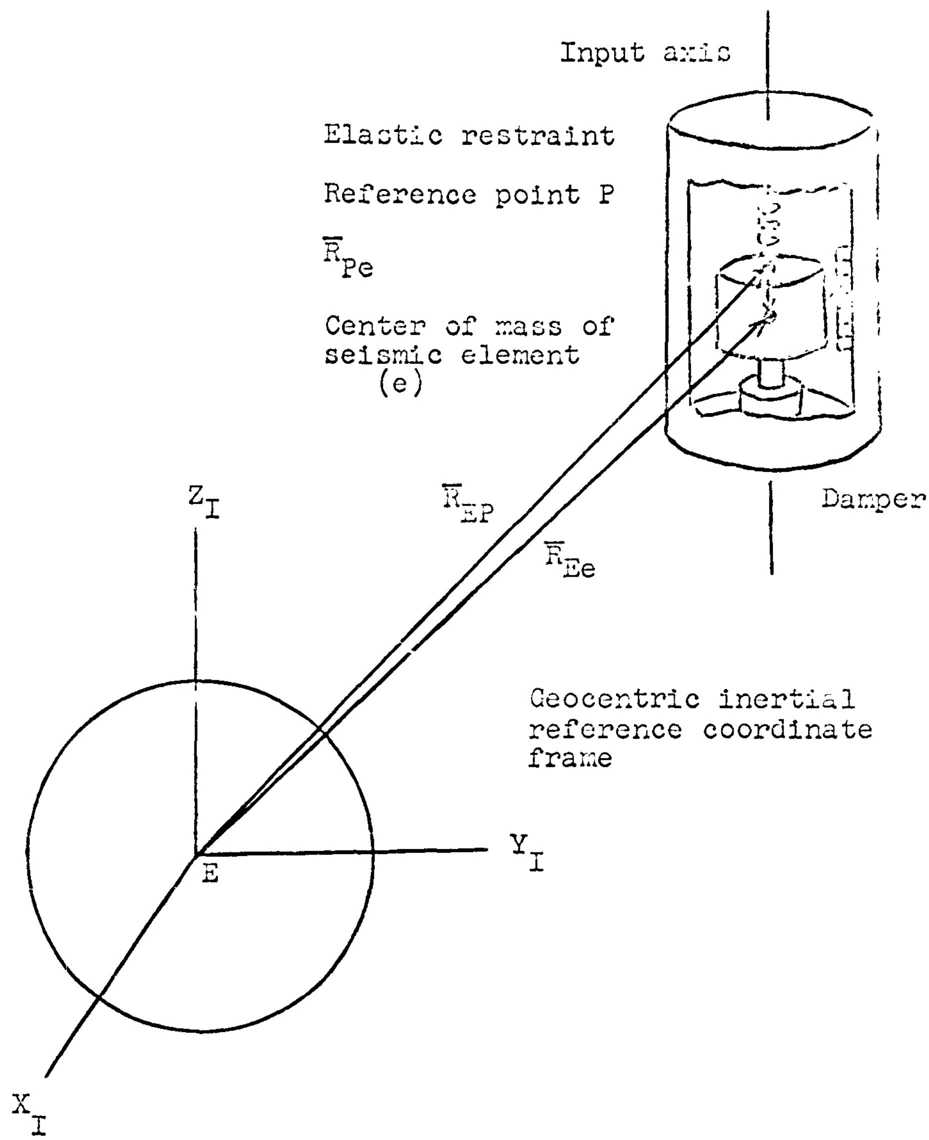

Figure 2-2

Vector Presentation Relating a Linear

Accelerometer to a Geocentric Inertial Reference Frame

$$= \frac{g}{R_e} R_{Pe} = \left[G_{EP} - (R_{EP})_I \right]_{ia} \qquad (2\text{-}9)$$

Any system which uses accelerometers to develop angular corrections to the vertical must start from some reference position and be maintained near some reference position in order to perform properly. This original reference must be put in as accurately as possible as initial conditions. The problem then becomes one of maintaining the reference as accurately as possible so that small deviations in vertical can be tracked precisely with the correction angles generated by the accelerometers. This task is performed through the use of gyros. The basic capabilities and characteristics of gyros are well documented, and will not be discussed in this thesis. The ability of present day gyros to maintain a fixed reference with respect to inertial space and with very low drift rates make the instrument the basic piece of hardware in any inertially referenced system.

The requirement placed on the system is that it maintain a three-dimensional coordinate frame for tracking the vertical while moving over the ellipsoidal surface of the rotating earth. If three accelerometers are placed with their input axes along the instrumented axis of a geographic coordinate frame, and the input axes of three single-degree-of-freedom integrating gyros are similarly located, the platform

can be maintained in its desired position through proper gimballing. Then the problem is to compute--automaticall and continuously--the angular velocity command signals for the three gyros so that, once aligned, the instrumented coordinates always tend to coincide with the local geographic coordinates (19). See Figure 2-3 for schematic diagram for such a three-axis stabilization system.

As stated before, the x and y accelerometers are used to determine quantities, which, when they are converted to angular rates, can be applies as corrections to the indicated vertical about the y and x axes, respectively. Since the indicated axes are rarely aligned with the geographic reference frame, the x and y axis accelerometers experience specific forces which have components along the reference vertical. Then, in order to compute the total specific force along the geographic vertical, the vertical components of the values sensed by the x and y accelerometers must be added to vertical component experienced by the z axis accelerometer.

A method for determining the total specific force component along the indicated z axis is derived by Broxmeyer (5). The indicated z axis is the actual vertical axis of a vertical indicating system, and is maintained parallel to the geographic vertical as

accurately as possible. Existing systems can keep this alignment within thirty seconds of arc, and possibly to even closer tolerances. The primary interest here is the specific force component in the z direction of the geographic reference frame. The closest information to this is the z component of specific force in the indicated geographic frame. For this reason, the following expression for this component in the indicated frame is given as:

$$(SF)_{z_i} = -(SF)_{x_g}\sin C_y + \left[(SF)_{y_g}\cos C_y\right]\sin C_x + \left[(SF)_{z_g}\cos C_y\right]\cos C_x \qquad (2\text{-}10)$$

where

$(SF)_{z_i}$ = total specific force along the indicated vertical

$$(SF)_{x_g} = g_{x_g} - \frac{d}{dt}\left[\dot{L}_g R_{EP}\cos D_N\right] - \dot{L}_g \frac{d}{dt}\left[R_{EP}\cos D_N\right] + \frac{d^2}{dt^2}\left[R_{EP}\sin D_N\right] - \dot{L}_g^{\,2} R_{EP}\sin D_N - \dot{\lambda} R_{EP}\left[2W_{IE} + \dot{\lambda}\right]\cos\left[L_g - D_N\right]\sin L_g$$

$$(SF)_{y_g} = g_{y_g} - \frac{d}{dt}\left[\dot{\lambda} R_{EP}\cos(L_g - D_N)\right] - \left[2W_{IE} + \dot{\lambda}\right]\frac{d}{dt}\left[R_{EP}\cos(L_g - D_N)\right]$$

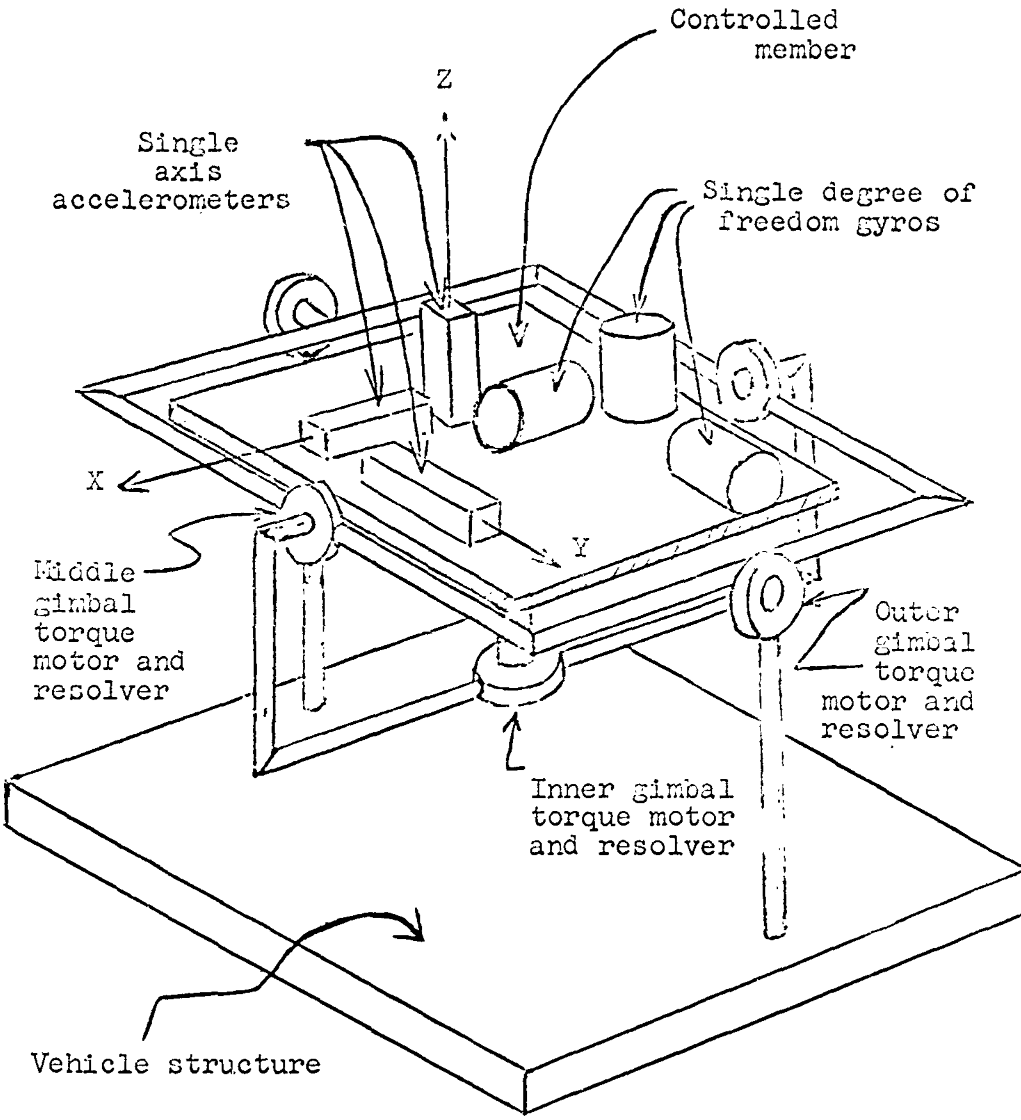

Figure 2-3

Three-axis Stabilized Platform

$$(SF)_{z_g} = g_{z_g} + \frac{d}{dt}\left[\dot{L}_g R_{EP} \sin D_N\right]$$

$$+ \dot{L}_g \frac{d}{dt}\left[R_{EP} \sin D_N\right] + \frac{d^2}{dt^2}\left[R_{EP} \cos D_N\right]$$

$$- \dot{L}_g^{\,2} R_{EP} \cos D_N - \dot{\lambda} R_{EP}\left[W_{IE} + \dot{\lambda}\right]$$

$$x \cos (L_g - D_N) \cos L_g$$

C_x and C_y = corrections to the indicated vertical about the x and y axes, respectively

Since the actual gravity normal to the geoid defines the astronomic vertical, and the geographic vertical is normal to the reference surface, only the deflection of the vertical separates the vertical we need from the geographic vertical. So, in order to put the above results into a more workable form, the magnitudes of the components of gravity along the geographic vertical will be transformed into the astronomic frame. Further simplification is possible through the small angle assumption. By assuming maintenance of the vertical to within 30 seconds of arc,

$$\cos C_x = \cos C_y \cong 1$$

$$\sin C_x = C_x$$

$$\sin C_y = C_y \qquad \text{(2-11)}$$

Since the deflection of the vertical averages from 15-30 seconds, and the largest known deflection is approximately one minute, the small angle assumption can again be made. Therefore,

$$g_{x_g} \cong D_{v_y}\, g$$

$$g_{y_g} \cong -D_{v_x}\, g$$

$$g_{z_g} \cong g \qquad (2\text{-}12)$$

where

D_{v_x} and D_{v_y} = deflections of the vertical about the x and y axes respectively.

Simplifying Equation (2-10) first by the relation of Equation (2-11) results in

$$(SF)_{z_1} = -\,(SF)_{x_g}\, C_y + (SF)_{y_g}\, C_x + (SF)_{z_g} \qquad (2\text{-}13)$$

Now, making the substitutions shown in Equation (2-12), a final expression for $(SF)_{z_1}$ is as follows:

$$(SF)_{z_1} = g\,(1 - C_x D_{v_x} - C_y D_{v_y}) + \ddot{L}_g R_{EP} (\sin D_N)$$

$$\times\,(1 + C_y) + 2\dot{L}_g \dot{R}_{EP} (\sin D_N)(1 + C_y)$$

$$+\, 2\dot{L}_g \dot{D}_N R_{EP} (\cos D_N)(1 - C_y)$$

$$+\, \ddot{R}_{EP} (\cos D_N)(1 - C_y) - 2\dot{R}_{EP} \dot{D}_N$$

$$\times (\sin D_N)(1 + C_y) - \ddot{D}_N R_{EP} (\sin D_N)(1 + C_y)$$

$$- \dot{D}_N^{\,2} R_{EP} (\cos D_N)(1 - C_y) - \dot{L}_g^{\,2} R_{EP} (\cos D_N)$$

$$\times (1 - C_y) - \left[\dot{\ell} R_{EP} (2W_{IE} + \dot{\ell}) \cos (L_g - D_N) \cos L_g \right]$$

$$\times (1 - C_y) - C_x \cos (L_g - D_N) \left[\ddot{\ell} R_{EP} + 2\dot{R}_{EP} \right.$$

$$\left. \times (W_{IE} + \dot{\ell}) \right] - 2C_x R_{EP} (W_{IE} + \dot{\ell}) \sin (L_g - D_N)$$

$$\times \left[L_g - D_N \right] \qquad (2\text{-}14)$$

We now have an exact expression which can be solved for the magnitude of the gravity vector. The accuracy with which we can find this value is based on the accuracy of the specific force sensor, and how well we know our position, velocity, longitude rate, latitude rate, etc. That is, any term which has an error will reflect an error in the final value of gravity. An analysis of these errors and their contribution to the overall error in g will be accomplished in Chapter 4 using the results of Chapter 3.

CHAPTER 3

NAVIGATION TECHNIQUES

The determination of the flight path projection on the surface of the earth is of primary importance. The solutions to this problem are categorized as follows (23):

1) Those systems which rely on measurements of the motion of the vehicle with respect to the medium in which it is traveling. Any motion of the medium with respect to the surface of the earth does, of course, cause errors.
2) Those systems which determine the position changes of a vehicle by measuring the acceleration of that vehicle. Inertial navigators are of this type. Both systems (1 and 2) are integrating systems. That is, one must integrate all measurements during flight in order to perform a navigation function. An interruption of data causes errors.
3) Cooperative systems in which the location of a vehicle is determined with the aid of established reference points upon the surface of the

earth--Loran, Tacan, etc. Systems of this class are capable of intermittent operation, since they are non-integrating.

4) Non-cooperative systems in which the location is determined by observing the relative movement of a reference--the surface of the earth or celestial bodies. Basically, these systems are of the integrating type, however, where particular reference points are identified (such as known elevations of particular stars), they are capable of non-integrating operation. Examples of this type are visual contact navigators, celestial navigators, Doppler radar navigators, and PECAN (Pulse Envelope Correlation Air Navigator). The PECAN system may be classified as an integrating type of radar sensing of a reference plane which makes use of information correlation, rather than Doppler frequency shifts.

The authors have decided that type 4 above is best suited for performing the required mission with sufficient accuracy. For this reason, we shall discuss the merits of the Doppler navigator and the Pulse Envelope Correlation Air Navigator.

In an airborne system to be used for measuring the value of gravity, four navigational quantities must be

known at all times:

1) Ground speed
2) Heading
3) Latitude
4) Longitude

The determination of these quantities can be broken into two classes:

1) Onboard measurements by the navigational system which determine the desired quantities directly, and
2) Postflight analysis to determine on a probabilistic basis an updated form of the information from the onboard measurements.

All information gathered by this system is in the time domain. It is transferred to the space domain when recorded through the use of a good clock.

The problem of navigation is best solved by restricting the flight to a straight line (constant true course) between two known points. While this may seem to be an idealization of any navigation problem, it will impose no operational restrictions, and will greatly enhance the probability of determining accurately the desired quantities.

The first problem is to determine accurately the terminal points of the flight path. It is proposed to do this by either radar or Loran C. Since all flights

for gravimetric purposes will begin and end reasonably near shore, either of these methods can be employed. Within twenty miles of a known coastal area, radar can give returns good enough to be considered free of error. The error should be on the order of a few hundred feet. When the gravity run is not to be started until out of radar range or over coastal areas where pinpoint radar returns are not possible, Loran C must be used. This gives an accuracy of 500 feet close to the station, and 1500 feet at a range up to 1300 nautical miles from the station (13). Neither of these can give accurate heading information. This must be obtained from the inertial platform.

During the flight, the inertial platform will be giving heading information to the autopilot. Current autopilots can hold headings to within 1/4 degree. The inertial platform will provide heading information sufficiently accurate to be considered free of error as compared with the 1/4 degree from the autopilot.[1]

Two methods are presented fro determining the ground speed and drift angle during the flight. The first, Doppler radar, is a well-established system, and is capable, with modifications, of giving an accuracy considered by the authors to be within the

1. According to Professor Whitaker of the M. I. T. Instrumentation Laboratory, these errors are what could be expected with state-of-the-art autopilots.

bounds necessary for airborne gravimetric measurements. The second, PECAN, is still under development, and is considered here for future application.

The APN-82, a Doppler system developed in 1957 and currently used by the Air Force, is capable of giving ground speed to ±2 knots, and drift angle to within .15° (1). For general navigation, these accuracies are well within the tolerances needed to satisfy the navigation problem. Several methods are presented to improve the Doppler accuracy for gravitational measurements.

As shown in Appendix B, the Doppler shift, f_d, is an average value for the frequency spectrum received.

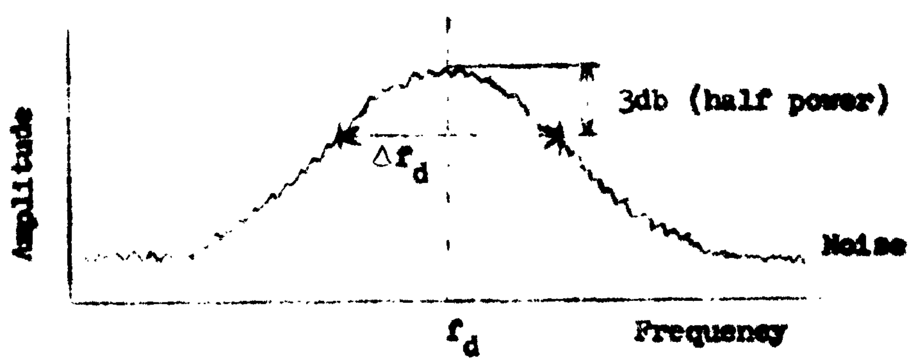

Figure 3-1

Typical Doppler Spectrum

$$\Delta f_d = \frac{2V}{\lambda} \sin\gamma\,\Delta\gamma \qquad (3\text{-}1)$$

Letting μ equal the mean, (f_d), it is advantageous to

examine the moments, μ_r, about the mean.

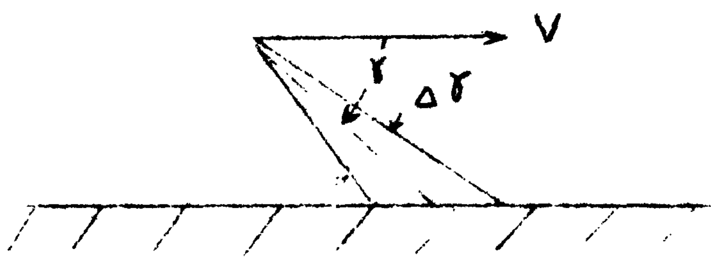

Figure 3-2

Doppler Geometry

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r \, p(x) \, dx \qquad (3\text{-}2)$$

where

μ_r = r th moment about the mean

μ = mean

x = a random variable

p(x) = probability of finding the value x

Here μ is obviously zero, and μ_2 is called the variance $\left[\sigma^2 (x)\right]$ of x. The third moment, μ_3, is a measure of the skewness of the distribution. For $\mu_3 < 0$, the curve tails to negative values of x. For $\mu_3 > 0$, the curve tails to positive values of x. Over land, μ_3 has been found experimentally to be zero. Over water, μ_3 will be positive and will vary in magnitude depending on the roughness of the sea.

The APN-82 has a sea mode which measures the mean (f_d) on an assumed skewness (μ_3) for an average sea state. For more accurate determination of ground speed, the APN-82 must be modified so that the change in μ_3 may be compensated. Figure 4 of Appendix B shows a variation in slope for variations in sea state (Beaufort numbers). The slope is a measure of the skewness of the Doppler frequency spectrum. From this, it is possible to eliminate average errors by putting the proper sea state conditions into the Doppl system. Figure 5 of Appendix B shows errors resulting in various sea states.

The Beaufort number could be obtained in the air and manually set into the Doppler system in either of two ways. First, the Beaufort number is a direct func of wind velocity. The Doppler system gives the wind t within 2% of its actual value (1). A second, and perhaps more accurate, method is to use a laser device, discussed later in this chapter, to measure the height of the sea waves. Such devices can measure distances with errors less than one feet. The width of the received spectrum (directly proportioned to wave height) can be used to determine Beaufort number.

Figure 3-1 shows the desirability of reducing μ_2 (the variance). As μ_2 decreases, the spectrum becomes more peaked. This will give a better deter-

mination of the average Doppler shift, f_d , with which to determine ground speed and drift. This can be improved by using larger antennas and hence a smaller beamwidth $\Delta \gamma$.

A second method to minimize μ_2 is to fly at a relatively low altitude. Over sea, the signal to noise ratio decreases as altitude increases. This drops the half power point (3 db) and increases Δf_d. When working with skewed distribution functions, an increase in Δf_d causes an increase in the uncertainty of finding the proper mean, f_d.

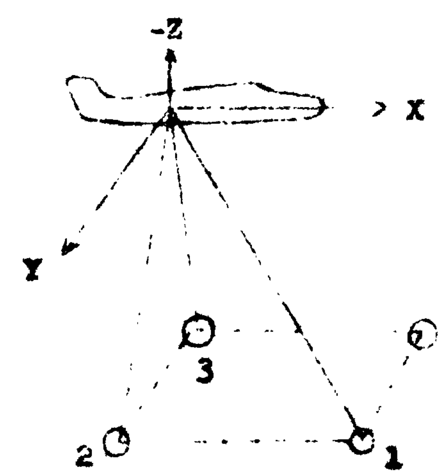

Figure 3-3

Janus Doppler System

$$V_x = \frac{\lambda}{4 \cos \alpha} (D_1 - D_3)$$

$$V_y = \frac{\lambda}{4 \cos \beta} (D_2 - D_3)$$

$$V_z = \frac{\lambda}{4 \cos \gamma} (D_1 + D_3) \qquad (3\text{-}3)$$

where

α, β, γ are angles between x and 1, y and 2, z and 3, respectively.

D_1, D_2, and D_3 are the Doppler shifts along beams 1, 2, and 3, respectively.

Figure 3-3 shows the configuration of a Janus type Doppler system. While the APN-82 has an accuracy of ±2 knots, newer systems are capable of accuracies of one knot at a ground speed of 200 knots. Tests of older Doppler radars (9) are shown in Appendix B to have the following errors:

1) Probable range error-- $8/\sqrt{D}$ %, where D is the distance traveled in nautical miles
2) Probable cross-course error-- $16/\sqrt{D}$ %
3) Probable position error-- $12/\sqrt{D}$ %

It is reasonable to expect accuracies improved on the order of 50% since the manufacturer's specifications have improved from 2 knots to 1 knot.

An alternate method for obtaining track and ground speed which may be employed in the future is based on a system which correlates variations in pulse radar terrain returns for airborne measurement of these quantities.

This system (PECAN) receives echoes at two antennas mounted along the longitudinal axis of an

aircraft (23). As the vehicle moves forward, the aft antenna assumes the previous position of the forward antenna. The two antennas thus receive identical return signals at a time differential determined by the aircraft velocity. The velocity information is obtained by measurement of the delay required for maximum correlation of the two signals.

The feasibility of a navigation system based on the correlation of returns to separated antennas of an incoherent pulse radar has been established. Such a system is insensitive to aircraft altitude, roll, and antenna position. Experimental data suggests that the system errors may be less than 1% (23). Several signal processing circuits have been explored in which the delay and correlation operations are accomplished with simple solid state electronic components. These may be useful in correlation devices other than PECAN.

It is pointed out that every airborne vehicle which carries a modern pulse-type radio altimeter has already the most important element of a PECAN system. A third antenna and a signal processing electronics package would be all that is required to provide velocity and ground track data.

In determining the position of the aircraft with respect to the earth, altitude is needed, as well as ground speed and track. Previous authors (21,25,26)

have relied on radar or barometric altimeters. The procedure suggested is to determine radar altitude over a known ground elevation (runway or lake) and then to measure deviations from this reference by using a barometric device. This method introduces large errors. Over the sea, it would not be necessary to use barometric devices, since the radar altimeter would measure height above the geoid directly. Although this appears to solve the problem, radar altimeters have inaccuracies on the order of ±30 feet in altitude determination at the proposed 10,000 foot altitude. As shown in Chapter 1, an error of 30 feet in altitude determination gives rise to a 3 mgal error.

A method to eliminate errors due to free air reductions is to use a laser radar device. This type of radar uses a small rod of synthetic ruby which absorbs light energy at one frequency, and emits light at another frequency. The light emitted is a brilliant, coherent ray which is capable of being pointed, with scattering much less than 1 milliradian. (11). The reflected light ray can then be timed to give altitude information in the same manner as radar devices. The important difference is that the extremely narrow beamwidth (1 milliradian) decreases the altitude error from ±30 feet to ± 1 foot. Since a one foot

error in altitude gives rise to a 0.1 mgal error in gravity, this error can effectively be disregarded.

If the onboard navigation system were perfect, a navigational fix, $x_f(t)$, at the end of the flight should coincide with the position indicated by the onboard navigation system. If they differ, a statistical method can be used to reduce the probable errors in ground speed, track, and position. This method uses the information obtained from the final positional fix of the flight to update all the information in ground speed, track, and position throughout the flight. Consider first the ground speed error.

Let $v(t)$ = actual velocity

$v_i(t)$ = velocity indicated by Doppler system

$\mathcal{E}(v)(t)$ = error in $v_i(t)$

$$v_i(t) = v(t) + \mathcal{E}(v)(t) \tag{3-4}$$

Choose $\bar{m}$ as the estimator for the average value of $\mathcal{E}(v)(t)$. The best estimate for $v(t)$ will then be

$$v(t) = v_i(t) + \bar{m} \tag{3-5}$$

Integrating Equation (3-4)

$$x_i(t) = \int_0^t v(\tau)\, d\tau + \int_0^t \mathcal{E}(v)(\tau)\, d\tau + x_i(0) \tag{3-6}$$

where

$x_i(t)$ = indicated position

$x_i(0)$ = initial position

$$x(t) = x(0) + \int_0^t v(\tau)\, d\tau \tag{3-7}$$

$$\mathcal{E}(x)(t) = \mathcal{E}(x(0)) + \int_0^t \mathcal{E}(v)(\tau)\, d\tau \tag{3-8}$$

$$x_i(t) = x(t) + \mathcal{E}(x)(t) \tag{3-9}$$

$$x_f(t) = x(t) + \mathcal{E}(x_f)(t) \tag{3-10}$$

where

$x_f(t)$ = measured final position

Define

$$\delta(x)(t) = x_i(t) - x_f(t) \tag{3-11}$$

$$= \mathcal{E}(x(0)) + \overline{\mathcal{E}(v)}\, t - \mathcal{E}(x_f)(t) + \int_0^t \epsilon(v)(\tau)\, d\tau \tag{3-12}$$

where

$\mathcal{E}(v)(t)$ = error in velocity

$$\mathcal{E}(v)(t) = \overline{\mathcal{E}(V)} + \epsilon(v)(t) \tag{3-13}$$

It may be assumed that $\epsilon(v)(t)$, the deviation from the mean of the error in velocity, is an unbiased, normally distributed, random variable. If $\overline{\mathcal{E}(V)}$ is

truly the mean of the error in velocity, the integral of $\in(v)(\tau)$ over t in the Equation (3-12) will be zero.

Since $\overline{\mathcal{E}(v)}$ is not known, it must be estimated. Equations (3-14) through (3-16) show that the estimator for the mean is actually unbiased. Let m be the estimator for the mean. The error in the estimate is $\overline{\mathcal{E}(v)} - m$. To make an unbiased estimate,

$$\bar{m} = \overline{\mathcal{E}(v)}$$

$$m = \frac{1}{t}\left[\delta(x) - \overline{\mathcal{E}(x(0))} + \overline{\mathcal{E}(x_f)}\right] \qquad (3\text{-}14)$$

$$\bar{m} = \frac{1}{t}\left[\overline{\delta(x)} - \overline{\mathcal{E}(x(0))} + \overline{\mathcal{E}(x_f)}\right] \qquad (3\text{-}15)$$

We have assumed

$$\overline{\in(v)} = \overline{\mathcal{E}(x(0))} = \overline{\mathcal{E}(x_f)} = 0$$

$$\bar{m} = \frac{1}{t}\left[\overline{\mathcal{E}(v)}\right]t = \overline{\mathcal{E}(v)} \qquad (3\text{-}16)$$

Therefore, m is an unbiased estimator for the mean of the error in velocity.

$$m = \frac{\delta(x)}{t} \qquad (3\text{-}17)$$

The variance of m is the quantity that determines the probability of an accurate measure of ground speed.

$$\overline{\left[\overline{\mathcal{E}(v)} - m^2\right]} = \overline{\left[\overline{\mathcal{E}(v)}^2 - 2\overline{\mathcal{E}(v)}m + m^2\right]} \qquad (3\text{-}18)$$

From Equation (3-9) it can be seen that $\overline{\mathcal{E}(v)^2}$, when averaged, will contain only one non-zero term,

$$\overline{\left[\frac{\delta(x)^2}{t}\right]}$$

From Equation (3-12), m^2, when averaged, has only one non-zero term,

$$\overline{\left[\frac{\delta(x)^2}{t^2}\right]}$$

The cross product term,

$$-2\,\overline{\mathcal{E}(v)\,m} = +2\,\overline{\frac{[\delta(x)]^2}{t^2}}$$

The variance, σ^2, is the sum of these three terms.

$$\sigma^2 = \frac{4}{t^2}\,\overline{[\delta(x)]^2} \tag{3-19}$$

Since there is only one measurement, $\delta(x)$,

$$\overline{[\delta(x)]^2} = [\delta(x)]^2$$

The standard deviation thus becomes

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{4\,\delta(x)^2}{t^2}} = \frac{2\,\delta(x)}{t} \tag{3-20}$$

The probable error in ground speed, $.674\,\sigma$, is then

$$PE_v = 1.348\,\frac{\delta(x)}{t} \tag{3-21}$$

It can be shown that the same equation is valid for determining the probable error in track.

CHAPTER 4

ANALYSIS OF ERRORS

Equation (2-14) provides a method by which readings from an accelerometer mounted on a stabilized platform in a moving vehicle can be converted into gravity measurements. The specific force measured along the indicated z axis can be corredted to a corresponding gravity value with the same error present in the accelerometer, in addition to the errors in the twelve correction terms in Equation (2-14). The effect of errors in these correction terms limits the accuracy with which gravity can be determined. To specify the errors anticipated in the gravity computation, it is first necessary for upper limits to be set on the operation of the system. In this thesis, the authors will examine the case of an aircraft flying at approximately 10,000 feet and 200 knots. It should be noted here that the optimum conditions for the purpose of this thesis would be operation at as low an airspeed as possible without losing acceptable autopilot performance. Higher ground speeds lead to larger coefficients in the correction terms as well as a decrease in accurate

determination of ground speed and its derivatives.

In the analysis performed in this chapter, the following values and assumptions were used:

1) Ground speed, 200 knots
2) Altitude, 10,000 feet
3) Aircraft always flown on autopilot with maximum excursions of ± 5 feet from reference altitude
4) Maximum values for errors instead of average errors were used throughout. The average error for any function is assumed to be zero, i.e., the equipment is unbiased
5) C_x and $C_y \leq 30 \widehat{sec} = 150 \times 10^{-6}$ radians
6) D_{v_x} and $D_{v_y} \doteq 15 \widehat{sec} \doteq 75 \times 10^{-6}$ radians

   There are a few isolated positions on the earth where this value reaches one minute of arc. However, these cases are rare, and 15 seconds is a good average
7) $R_{EP} = 20.9 \times 10^6$ feet $= 6.44 \times 10^8$ cm.
8) $\mathcal{E}(R_{EP}) = \pm 1$ foot
9) $D_N \leq 11.5 \widehat{min.} = 3.34 \times 10^{-3}$ radians

   (maximum value at $L_g = 45°$)
10) $\mathcal{E}(L_g) = \mathcal{E}(\ell) = 1$ minute of arc
11) $\dot{L}_g \leq 200$ NM/hr $= 1.62 \times 10^{-5}$ rad/sec

12) $\epsilon(\dot{L}_g) = 1\%\ \dot{L}_g = 2$ nm/hr $= 1.62 \times 10^{-7}$ rad/sec

13) $\dot{\ell} = \dfrac{200 \text{ nm/hr}}{\cos(L_g - D_N)} = \dfrac{1.62 \times 10^{-5} \text{ rad/sec}}{.342}$

$= 4.75 \times 10^{-5}$ rad/sec

14) $\epsilon(\dot{\ell}) = 1\%\ \dot{\ell} = 4.75 \times 10^{-7}$ rad/sec

15) $\dot{R}_{EP} = 10$ ft/min $= .167$ ft/sec

16) $\epsilon(\dot{R}_{EP}) = .0167$ ft/sec (altimeter is accurate to $\pm$ 1 foot as examined in Chapter 2)

17) $\dot{D}_N = 1.08 \times 10^{-11}$ rad/sec ($\dot{D}_N$ is a maximum at $L_g = 0$. $D_N = (11.5 \text{ min}) \sin 2L_g$; and $\dot{L}_g$ max $= 1.62 \times 10^{-5}$ rad/sec

18) $\epsilon(\dot{D}_N) = 1.08 \times 10^{-9}$ rad/sec

(from error in ground speed)

19) $\ddot{D}_N = 2.32 \times 10^{-12}$ rad/sec^2 (at 200 knots it takes $45° \Big/ 3.33°/\text{hrs} \times 3600 \ \frac{\text{sec}}{\text{hr}}$

$= 4.85 \times 10^{-4}$ sec to travel 45°.

1.08×10^{-9} rad/sec divided by 4.85×10^{4} sec

$= 2.32 \times 10^{-12}$ rad/sec)

20) $\mathcal{E}(\ddot{D}_N) \doteq 0$

21) $\ddot{R}_{EP} = 0$ (average over an integral number at

phugoid periods as discussed in Chapter 2)

22) $W_{IE} = 7.29 \times 10^{-5}$ rad/sec

23) L_g (maximum) has been arbitrarily set at

70° latitude to prevent excessive values of

$\dot{\gamma}$ at 200-knot ground speed

24) $\ddot{L}_g(\max) = .2g = \dfrac{a(\max)}{R_{EP}} = \dfrac{.20 \times 32.2 \text{ ft/sec}^2}{20.9 \times 10^6}$

$= 3.08 \times 10^{-7}$ rad/sec

25) $\mathcal{E}(\ddot{L}_g) = \dfrac{a(\max)}{R_{EP}} = \dfrac{.67 \times 10^{-3} \text{ ft/sec}^2}{20.9 \times 10^6 \text{ ft}}$

$= 3.2 \times 10^{-11}$ rad/sec^2

26) $\ddot{\gamma}(\max) = \dfrac{\ddot{L}_g(\max)}{\cos(L_g - D_N)} = \dfrac{3.08 \times 10^{-7}}{.342}$

$$= 9 \times 10^{-7} \text{ rad/sec}^2$$

$$27) \quad \mathcal{E}(\bar{\gamma}) = \frac{(\ddot{L}_g)}{\cos\left(L_g(\max) - L_N\right)} = \frac{3.2 \times 10^{-11}}{.342} \text{ rad/sec}^2 = 9.35 \times 10^{-11} \text{ rad/sec}^2$$

Since the aircraft will be flown under the most ideal conditions possible (straight, level and unaccelerated flight on autopilot), the maximum sustained acceleration in the horizontal direction is assumed to be .2g (6,26). Furthermore, transient disturbances are assumed to be of short duration, and to average to zero over the four to five minute interval of interest. As previously discussed, this assumption is justified through the use of an undamped Schuler tuned platform, and by the measurement being made over an integral number of aircraft phugoidal periods.

$\ddot{L}_g$ can be found in a number of ways. Differentiation of ground speed as determined by the Doppler set provides one method. This operation is difficult, and accuracies are poor due to the noise present in the return signal. This method has been by-passed by the authors in favor of a more precise technique. A second and more accurate method would be to use a horizontal accelerometer. The only sacrifices in accuracy here would be due to the inherent capability of the

accelerometer and the inability of the stable platform to maintain the input axis horizontal. This last problem is mostly due to the drift present in the gyros used in the stable platform. Another small error here is due to the inaccurate representation of the aircraft heading (assumed $\pm$ 1/4 of a degree). These errors may combine to make this process unusable for the purposes of this thesis. However, as the quality of gyros and accelerometers is improved, this may well become the most accurate method available.

A third method for determining the value of $\dot{L}_g$ seems to be the most accurate means available at present. This process entails the logical assumption that velocity errors are propagated at the Schuler tuning frequency of 84.4 minutes. This is not merely an assumption, but has been proven on test flights at the M. I. T. Flight Test Facility. If the maximum error in velocity is known, or can be found from post-flight analysis, it is a simple process to determine the maximum rate of change of velocity error. The assumption that this maximum velocity error is less than one foot per second gives the system the capability desired for this thesis. Figure 4-1 shows that the maximum rate of change of velocity error occurs during the 1/4 cycle centered at a zero velocity error crossing. Since 1/4 period equals 21.1 minutes, the

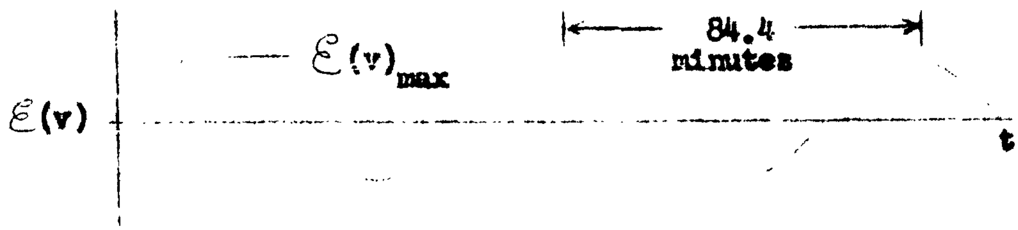

Figure 4-1

Velocity Errors in an Inertial System

$$\mathcal{E}\,a_{(max)} = \frac{\mathcal{E}(v_{max}) \times 1.414}{21.1 \times 60} \quad (ft/sec^2).$$

If the maximum velocity error is assumed to be one foot per second, then the $\mathcal{E}\,a_{(max)}$ 1.0×10^{-3} ft/sec^2. This value, when divided by R_{EP}, gives 3.2×10^{-11} rad/sec^2 as $\mathcal{E}\,(\ddot{L}_g)$.

Table 4-1 summarizes the errors (in mgal) in the measurement of gravity using the above errors in Equation (2-14). Table 4-1 shows that the major contributing factor to errors in the measurement of gravity is the inability of the system to measure ground speed, specifically, $\dot{\ell}$. This error is reduced from ± 4.2 mgal by a factor of $1/\cos\,(L_g - D_N)$. At the equator, the error due to error in $\dot{\ell}$ reduces to ± 1.4 mgal. This is a basic limitation on the system proposed by the authors. At high latitudes, the error

TABLE 4-1 ERRORS IN SPECIFIC FORCE MEASURED ALONG THE Z AXIS

	TERM IN EQUATION (2-14)	APPROXIMATE ERROR IN MGAL
1	$(SF)_{z_i}$	± 1.5
2	$g(1 - C_x D_{v_x} - C_y D_{v_y})$	$\pm 2.25 \times 10^{-2}$
3	$\ddot{L}_g R_{EP} (\sin D_N)(1 + C_y)$	$\pm 6.9 \times 10^{-2}$
4	$2\dot{L}_g \dot{R}_{EP} (\sin D_N)(1 + C_y)$	$\pm 1.66 \times 10^{-3}$
5	$2\dot{L}_g \dot{D}_N R_{EP} (\cos D_N)(1 - C_y)$	$\pm 2.25 \times 10^{-6}$
6	$\ddot{R}_{EP} (\cos D_N)(1 - C_y)$	Assumed to be zero
7	$2\dot{R}_{EP} \dot{D}_N (\sin D_N)(1 + C_y)$	$\pm 1.11 \times 10^{-11}$
8	$\ddot{D}_N R_{EP} (\sin D_N)(1 + C_y)$	$\pm 2.38 \times 10^{-11}$
9	$\dot{D}_N^2 R_{EP} (\cos D_N)(1 - C_y)$	$\pm 3.6 \times 10^{-18}$
10	$\dot{L}_g^2 R_{EP} (\cos D_N)(1 - C_y)$	$\pm .386$
11	$\dot{\ell} R_{EP} (2W_{IE} + \dot{\ell}) \cos (L_g - D_N) \times (\cos L_g)(1 - C_y)$	± 4.2
12	$C_x \cos (L_g - D_N) \left[\ddot{\ell} R_{EP} + 2\dot{R}_{EP} (W_{IE} + \dot{\ell})\right]$	$\pm 3.08 \times 10^{-3}$
13	$2C_x R_{EP} (W_{IE} + \dot{\ell}) \sin (L_g - D_N) \times \left[L_g - D_N\right]$	$\pm 1.5 \times 10^{-3}$
14	Free air correction error	$\pm .1$

in the measurement of gravity exceeds the desired 3 to 4 mgal range when flying an east-west course. The obvious solution for gravity measurements at high latitudes is to fly a north-south course during measurement runs. Note also that the error in term 10 due to an error in $\dot{L}_g$ cannot be at a maximum when the error in term 11 due to an error in $\dot{l}$ is at a maximum. Two rms values of error in the measurement of gravity are given below--one for a north-south course, and one for an east-west course. The errors are due to errors in terms 1, 10, 11, and 14. The remaining terms have errors less than .1 mgal, and may be disregarded.

NORTH-SOUTH COURSE

$$\text{rms error} = \sqrt{(1.5)^2 + (.4)^2 + (.1)^2}$$

$$= 1.56 \cong 1.6 \text{ mgal}$$

EAST-WEST COURSE

$$\text{rms error} = \sqrt{(1.5)^2 + (4.2)^2 + (.1)^2}$$

$$= 4.46 \cong 4.5 \text{ mgal}$$

CHAPTER 5

SUMMARY AND CONCLUSIONS

The mechanization of the theory and techniques discussed in previous chapters is not impossible or even difficult. However, there are certain requirements of this mechanization which must be considered in detail. The selection of a suitable aircraft to perform the mission must be made. Some of the desired characteristics of the chosen aircraft are as follows:

a) Sufficient range to span all ocean masses
b) Stable flight characteristics in desired airspeed range with good autopilot, i.e., one capable of maintaining altitude within ± 20 feet, and heading within $\pm 1/4$ degree
c) Capable of mounting all equipment associated with gravity measurement
d) Phugoidal period known at desired airspeed--actually, this is not a function of the aircraft, but a function of the airspeed. However, the magnitude of the excursions from the reference altitude is a function of the aircraft configuration. The phugoidal period of an aircraft flying at 200 knots is approximately 46.5 seconds. (Since $w_{n(\text{phugoidal})} = \frac{g}{V}\sqrt{2}$

an increase in airspeed decreases w_n and thereby increases the phugoidal period.)

The world-wide gravimetric requirements as outlined by Dr. Lloyd Thompson of the Geophysics Research Directorate, Air Force Cambridge Research Center, set forth the need for average gravity values over a 60 mile interval. For an aircraft flying at 200 nm/hr, an 18-minute gravity reading could be made to obtain this average. The random errors which are assumed to average to zero over a gravity measurement would probably average to zero over a much shorter time interval--probably four to five minutes (20,24,25). During this time, the aircraft travels over approximately fifteen miles and goes through six phugoidal periods. This number of phugoidal periods is believed to be sufficient to average the long period vertical acceleration. In seaborne gravimeters, no such ability to measure long period vertical acceleration exists due to the lack of an external reference. Still, experiments in this type of gravity measurement indicate that five minutes is long enough for an adequate averaging (4,7,16).

Any errors resulting from disturbances which do not average to zero can be minimized by flying the mission in the best possible weather conditions. The effects of any horizontal disturbances are included in

the error analysis already presented.

The information from the accelerometer, stabilized platform, altimeter, and Doppler radar will be recorded in the time domain by the use of a clock. The measured information in time can then be analyzed on the ground and converted back to a space (i.e., position) function. The mechanics of averaging all the inputs to Equation (2-14) is best done utilizing a digital computer. The same computer can be used to solve Equation (2-14) using the average values as calculated by the computer. Previous authors (7,21, 24,25,26) have found that reduction of data without the use of a computer is an extremely tedious task.

The use of Loran C would permit the aircraft to accomplish the mission without being outside of Loran coverage for periods in excess of five hours. To prevent the longitude rate from becoming excessive, missions flown along parallels of latitude should be limited to latitudes less than 60°. For areas above this latitude, tracks along meridians close enough together to provide average readings over 60 nautical mile squares presents one method of measurement in the higher latitudes.

The present world-wide coverage of Loran C is insufficient for this mission. However, if all present Loran stations are replaced with Loran C, there will be

no problem. This replacement should be complete within the next few years. There are inertial systems in existence which will give the required accuracy. So, it is possible to do the job today.

The inertial system possesses other advantages. One is the fact that ocean currents do not reduce its accuracy. Ocean currents are approximately known, and will not appreciably deteriorate the ground speed determined by Doppler. However, it does not enter into the inertial system at all. Another advantage of the inertial system would be the following: Since any error in position and velocity propagates with the Schuler period in an inertial system, and remains fairly constant in amplitude for each flight, a calibration period of approximately 42 minutes (1/2 Schuler period) can be used to calibrate the error out of the system for each run. In other words, we can find out what the error in position and velocity is for each run. With post-flight analysis, these errors can therefore be removed from the readings taken. This technique provides a large reduction in the errors presented in the error analysis chapter. The maximum error in gravity measurement while flying along parallels of latitude up to 60° becomes less than 2 mgal. Maximum error flying along meridian also becomes less than 2 mgal.

Specific System

Aircraft--C-97 or C-121 with autopilot. These two aircraft meet previously established criteria.

Equipment--

a) Laser Altimeter

b) PIGA or PIPA for use as gravity measuring device.

c) Inertial system which will accurately maintain the reference vertical (the normal to the reference ellipsoid) and stable platform on which the gravity meter is placed. The inertial package will also provide the following:

1) Aircraft heading reference

2) Stabilizing signals for the Laser altimeter antenna

d) Loran C equipment

e) Doppler APN-82

f) Readout and recording equipment for the gravimeter.

Future Possibilities

The next few years should provide equipment far superior to that assumed in this thesis, according to Mr. Polner of the M. I. T. Flight Test Facility, and Mr. Sciegienny of the Instrumentation Laboratory.

The inertial type system will be able to determine position to within a quarter of a mile after five to ten hours of operation, and velocities to within one to two feet per second. These improvements, coupled with a specific force measuring device with a 50% improvement over what can be done today, will permit the techniques discussed here to be utilized with very high accuracy.

Continuous-laser Doppler equipment, when developed, will permit accuracy very close to inertial system quality at a much lower cost, according to Mr. Sciegienny. This possibility may not be realized immediately; however, its applicability in this scheme is quite evident if its development is as successful as presently expected.

The two possible future developments mentioned above are not the only means of improving the scheme discussed here. An integrated system containing inertial, Doppler, and laser equipment can be utilized to provide better accuracy with state-of-the-art components. The equations and constants for such a system were developed by Mr. Sciegienny of the M. I. T. Instrumentation Laboratory. This integrated system contains third order damping, and utilizes accurate altitude and a properly filtered Doppler signal to correct the inertial system. Such a system would

also be improved through the use of a continuous-laser type Doppler radar.

Present System Errors

The errors of Chapter 4 are derived from a system employing Doppler radar to determine ground speed. As was indicated previously, these errors can be reduced considerably (to less than 2 mgal) by the use of an inertial system alone. A system to do this, however, is still in its testing stages. This is the reason for employing Doppler, even though its accuracy is less than the inertial system.

Proper planning of missions to be flown will reduce the maximum rms error to acceptable values. If a north-south course could be flown throughout, the rms value of error would have a maximum of 1.6 mgal. This is not feasible because the aircraft would be flying for extended periods outside areas where its position could be determined with sufficient accuracy. There is a method for reducing the east-west course error to 4 mgal or less; this consists of limiting the latitude for such courses to 60°. Above this latitude, north-south courses can be flown with rms error = 1.6 mgal. This error satisfies the requirements originally specified. This thesis has shown theoretically that an aircraft carrying state-of-the-art equipment can make an over-water gravimetric

survey with sufficient accuracy to improve the present estimate for the shape of the earth.

APPENDIX A

STOKES' THEOREM

Stokes' Theorem states that the shape of the earth can be determined through the use of variations of the value of gravity about a reference value. His method relates a differential in gravity to a corresponding differential in distance. Thus, if the reference is know (1,2,30), the actual shape of the earth can be obtained by deviations from the reference frame.

The equation for δq, as developed by Heiskanen and Vening-Meinesz (2) is

$$\delta q = \sum_{n=2}^{n=\infty} \frac{2n+1}{4\pi} \int_{o}^{\sigma} P_n \ \delta q \ d\sigma \tag{A-1}$$

where

P_n = a zonal spherical harmonic in ψ, the angle between the radius vector of the external point at which we wish to determine N_ρ and the element of the geoid where the gravity anomaly is δg.

$d\sigma$ = element of a sphere of unit radius with a radius vector coinciding with that of δg.

If δg and N_o are set up in spherical harmonics Y,

$$\delta g = \frac{Y_2}{r^4} + 2\frac{Y_3}{r^5} + 3\frac{Y_4}{r^6} + \ldots + (n+1)\frac{Y_n}{r^{n+2}} \tag{A-2}$$

and

$$N_o = \frac{r}{g}\left[\frac{Y_2}{r^4} + \frac{Y_3}{r^5} + \frac{Y_4}{r^6} + \ldots + \frac{Y_n}{r^{n+2}}\right] \tag{A-3}$$

and the summation and integration are interchanged in Equation (A-1),

$$N_\rho = \frac{\rho}{4\pi g_\rho}\int_o^\sigma \sum_{n=2}^{n=\infty} \frac{2n+1}{n-1} P_n \left(\frac{r}{\rho}\right)^{n+2} \delta g d\sigma \tag{A-4}$$

If the function

$$F_\rho = \sum_{n=2}^{n=\infty} \frac{(2n+1)}{n-1} P_n \left(\frac{r}{\rho}\right)^{n-1} \tag{A-5}$$

is introduced, Equation (A-4) becomes

$$N_\rho = \frac{r^3}{4\pi g_\rho \rho^2}\int_o^\sigma F_\rho \delta g d\sigma \tag{A-6}$$

If the notation

$$q = \sum_{n=2}^{n=\infty} P_n \left(\frac{r}{\rho}\right)^n \quad \text{and} \quad u = \frac{r}{\rho}$$

is introduced, it can be shown that

$$q = \sum_{n=2}^{n=\infty} P_n u^n = (1 - 2u \cos\psi + u^2)^{-\frac{1}{2}}$$

$$- P_0 - P_1 u$$

$$= (1 - 2u \cos\psi + u^2)^{-\frac{1}{2}} - 1 - u \cos\psi \qquad \text{(A-7)}$$

Then

$$F_\rho = \sum_{n=2}^{n=\infty} \frac{2n+1}{n-1} P_n u^{n-1} = \sum_{n=2}^{n=\infty} \int_0^u (2n + 1)$$

$$\times P_n u^{n-2} d u$$

$$= 2 \int_0^u u^{-\frac{3}{2}} \frac{d}{du} \left[\sum_{n=2}^{n=\infty} P_n u^{n+2} \right] d u$$

$$= 2 \int_0^u u^{-3/2} \frac{d}{du} (u^{1/2} q) d u \qquad \text{(A-8)}$$

If the equation for q (A-7) is substituted in (A-8), and the quantity $v = (1 - 2u \cos\psi + u^2)^{1/2}$ is introduced,

$$F_\rho = \frac{2}{uv} + \frac{1}{u} - 5 \cos\psi - 3 \frac{v}{u}$$

$$- 3 \cos\psi \ \ln 1/2 (1 - u \cos\psi + v) \qquad \text{(A-9)}$$

It is relatively easy to compute values of F_ρ for all values of ρ and ψ. From (A-6) all values of N_ρ may be calculated. If we take $\rho = 0$ as the surface

of the geoid, we have $u = 1$, $v = 2 \sin 1/2\psi$ and $F_o =$
$\csc 1/2\psi + 1 - 5 \cos\psi - 6 \sin 1/2\psi - 3 \cos\psi \ln\left[\sin 1/2\psi\right.$
$\left. x \ (1 + \sin 1/2\psi)\right]$.

F_o is seen to be simply a mathematical function of ψ, the angle between the radius vector at which we wish to determine N_o and the element to the geoid where the gravity anomaly is δg. Thus, Equation (A-6) becomes

$$N_o = \frac{1}{4\pi g r} \int_0^S F_o \, \delta g dS \qquad (A\text{-}10)$$

where g and r are mean values over the geoid and S is a sphere of mean radius r. This is the theorem of Stokes, used for computing the shape of the geoid.

APPENDIX B

DOPPLER NAVIGATION ERRORS

This appendix deals with the mathematical analysis of the probable error and its distribution. The error standard deviation for the general case will be derived (27).

Let the straight line of distance D be the course between departure and destination. The errors in ground speed and track will be designated X and Y respectively (Figure B-1). The course is aligned

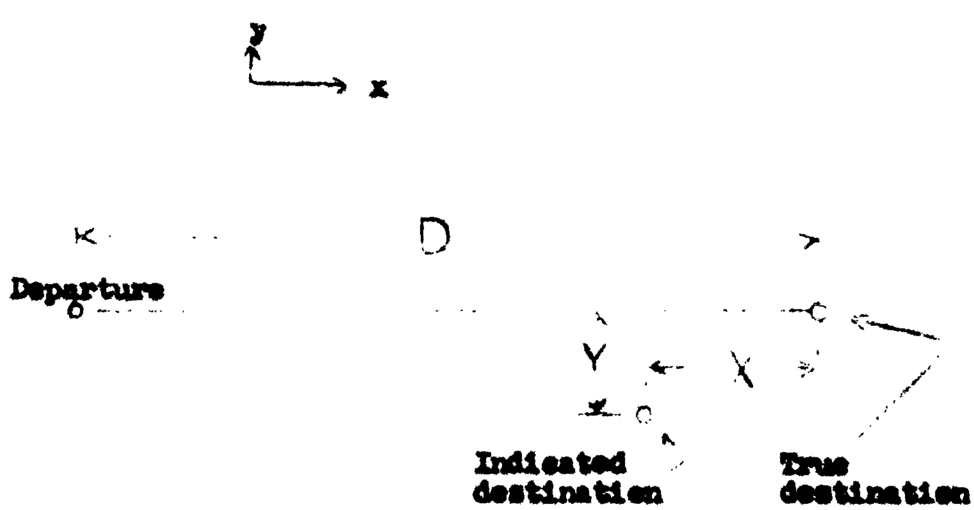

Figure B-1

Definition of Errors

with the X axis. Reference to the along-course errors will be made with the subscripts ao, and reference to the across-course errors (cross-course) is made with the subscripts cc. The indicated errors X and Y are cumulative errors, and can best be studied by statistical methods by dividing the distance D into equal segments ΔD so that the complete course is divided into n equal segments. Each segment is sufficiently long so that the errors attributed to each segment are statistically independent. Any errors from previous segments will in no way influence errors generated in future segments. It is assumed, in this derivation, that constant errors can be biased or calibrated out.

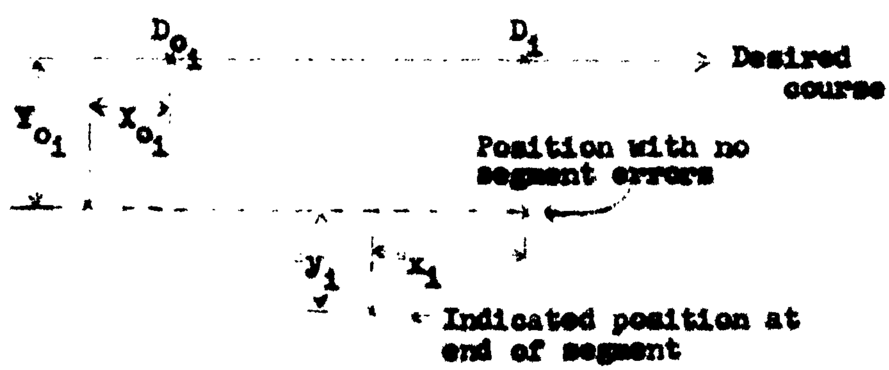

Figure B-2

Definition of Segment Errors

If x is a variable, an error in x for the i th observation will be noted as Δx_i. The mean value of Δx_i for n observations will be noted as $\overline{\Delta x_i}$

where

$$\overline{\Delta x_i} = \frac{1}{n} \sum_{i=1}^{n} \Delta x_i \tag{B-1}$$

The deviations from the mean, the absolute value of $(\Delta x_i - \overline{\Delta x_i})$ describe the scattering of the observations about the mean, and these are described by means of the variance $\sigma_x^{\ 2}$, where

$$\sigma_x^{\ 2} = \frac{1}{n} \sum_{i=1}^{n} (\Delta x_i - \overline{\Delta x_i})^2 \tag{B-2}$$

A more useful form of this equation is

$$\frac{1}{n} \sum_{i=1}^{n} \Delta x_i^{\ 2} = \sigma_x^{\ 2} + \overline{\Delta x_i}^{\ 2} \tag{B-3}$$

The variance may also be described as the second-order moment of the distribution. If the shape of the distribution curve is known, then the mean and the variance describe the particular distribution fully. The standard deviation σ_x, which is the positive square root of the variance, is frequently used as a more meaningful term for distribution of errors.

Calling m the number of segments $D/\Delta D$ into which the course is divided, the cumulative errors X and Y at the end of the flight are given by

$$X = \sum_{i=1}^{m} \Delta x_i \tag{B-4}$$

$$Y = \sum_{i=1}^{m} \Delta y_i \tag{B-5}$$

Thus, the cumulative error as obtained by the navigation system is a linear function of random variables.

If a number of flights are analyzed, the errors of each flight will be independent of errors in other flights. Thus,

$$X = \sum_{i=1}^{m} \overline{\Delta x_i} \tag{B-6}$$

$$Y = \sum_{i=1}^{m} \overline{\Delta y_i} \tag{B-7}$$

and

$$\sigma_X^{\ 2} = \sum_{i=1}^{m} \sigma_{\bar{x}_i}^{\ 2} \tag{B-8}$$

$$\sigma_Y^{\ 2} = \sum_{i=1}^{m} \sigma_{\bar{y}_i}^{\ 2} \tag{B-9}$$

However, since every segment has the same basic characteristics,

$$\overline{\Delta x_n} = \overline{\Delta x_{n+1}} = \overline{\Delta x_{n+2}} \quad \text{etc.} \tag{B-10}$$

$$\overline{\Delta y_n} = \overline{\Delta y_{n+1}} = \overline{\Delta y_{n+2}} \quad \text{etc.} \tag{B-10}$$

$$\sigma_{x_n}^{2} = \sigma_{x_{n+1}}^{2} = \sigma_{x_{n+2}}^{2} \quad \text{etc.}$$

$$\sigma_{y_n}^{2} = \sigma_{y_{n+1}}^{2} = \sigma_{y_{n+2}}^{2} \quad \text{etc.} \tag{B-11}$$

the summations can be carried out using the i th subscript to denote the value of the mean and variance for the whole family of observations,

$$\bar{X} = m\ \overline{\Delta x_1} = \frac{D}{\Delta D}\ \overline{\Delta x_1} = D\,\Delta x \tag{B-12}$$

$$\bar{Y} = m\ \overline{\Delta y_1} = \frac{D}{\Delta D}\ \overline{\Delta y_1} = D\,\Delta y \tag{B-13}$$

where

$$\overline{x_1}/\Delta D = \Delta x$$

$$\overline{y_1}/\Delta D = \Delta y$$

Similarly,

$$\sigma_X^{2} = m\,\sigma_{x_1}^{2} = D/\Delta D\ \sigma_{x_1}^{2} = D\,\sigma_x^{2} \tag{B-14}$$

$$\sigma_Y^{2} = m\,\sigma_{y_1}^{2} = D/\Delta D\ \sigma_{y_1}^{2} = D\,\sigma_y^{2} \tag{B-15}$$

where

$$\sigma_{x_1}^{2}/\Delta D = \sigma_x^{2}$$

$$\sigma_{y_1}{}^2 / \Delta D = \sigma_y{}^2$$

The deviations are the square roots of equations (B-14) and (B-15).

$$\sigma_X = \sqrt{D}\,\sigma_x \qquad \text{(B-16)}$$

$$\sigma_Y = \sqrt{D}\,\sigma_y \qquad \text{(B-17)}$$

In order to obtain the mean error per mile, the cumulative mean errors $\bar{X}$ and $\bar{Y}$ are divided by the distance D to obtain Δx and Δy for the mean error per mile, and

$$\sigma_{ac} = \frac{\sigma_X}{\sqrt{D}} \qquad \text{(B-18)}$$

$$\sigma_{cc} = \frac{\sigma_Y}{\sqrt{D}} \qquad \text{(B-19)}$$

are the along-course and cross-course deviations.

These values have been obtained for a single family of observations which means that the longer the distance, the closer that the error will approach the mean error as a percentage of distance traveled. The mean error is proportional to distance, the length of the trip, for a family of observations.

To obtain the error distribution to be used in

predicting accuracies in the use of all Doppler navigation systems, it is necessary to consider a whole population of navigation runs under many different conditions. Calling the j th subset and designating it by a j subscript

mean error = $\bar{X}_j$ and $\bar{Y}_j$

$$\text{deviation} = \frac{\sigma_{x_j} D_j}{\sqrt{D}} \quad \text{and} \quad \frac{\sigma_{y_j} D_j}{\sqrt{D}}$$

$$= \sigma_{x_j}\sqrt{D_j} \quad \text{and} \quad \sigma_{y_j}\sqrt{D_j} \tag{B-20}$$

This means that each error as defined by (B-4) will be made up of a mean error plus a deviation from the error, and

$$X_j = \bar{X}_j + \Delta X_j \quad \text{and}$$

$$Y_j = \bar{Y}_j + \Delta Y_j \tag{B-21}$$

The distribution of ΔX_j and ΔY_j is determined by the deviation values given in (B-20). The mean error is determined by summing up (B-21) over the population n and inasmuch as now every possible variable factor is allowed to vary, there are no

reasons for any mean value to exist unless it is built into the equipment and this, of course, is not done. Substituting (B-12) and (B-13) into (B-21),

$$X_j = D_j \triangle x_j + \triangle X_j$$

$$Y_j = D_j \triangle y_j + \triangle Y_j \qquad \text{(B-22)}$$

where $\triangle x_j$ and $\triangle y_j$ are constants determined by the state of the equipment and all other variable conditions during the j th run. Substituting (B-22) into the variance equation, (B-2),

$$\sigma_{X_j}^{2} = \frac{1}{n} \sum_{j=1}^{n} (D_j \triangle x_j + \triangle X_j)^2 \qquad \text{(B-23)}$$

and

$$\sigma_{Y_j}^{2} = \frac{1}{n} \sum_{j=1}^{n} (D_j \triangle y_j + \triangle Y_j)^2 \qquad \text{(B-24)}$$

again, since these are the variances of a linear combination of variables. For simplicity of calculations, let $D = D_j$ = any value

$$\sigma_{Y_j}^{2} = \frac{1}{n} \sum_{j=1}^{n} D^2 \triangle y^2_j + \frac{1}{n} \sum_{j=1}^{n} \triangle Y_j^{2} \qquad \text{(B-25)}$$

and

$$\sigma_{X_j}^2 = \frac{1}{n}\sum_{j=1}^{n} D^2 \Delta x_j^2 + \frac{1}{n}\sum_{j=1}^{n} \Delta X_j^2 \qquad \text{(B-26)}$$

Since $\sigma_{x_j}^2$ and $\sigma_{y_j}^2$ are multiplied by a constant D^2, the D^2 can be taken outside the summation constant and letting

$$\sigma_{ac'}^2 = \frac{1}{n}\sum_{j=1}^{n} \Delta x_j^2 \qquad \text{(B-27)}$$

$$\sigma_{cc'}^2 = \frac{1}{n}\sum_{j=1}^{n} \Delta y_j^2 \qquad \text{(B-28)}$$

In addition, the second term of (B-25) and the second term of (B-26) are merely the definition of variance which is the square of the deviation as given in (B-20). Thus, substituting (B-20), (B-27), and (B-28) into (B-25) and (B-26), also using (B-18) and (B-19),

$$\sigma_{Y_j}^2 = D^2 \sigma_{cc'}^2 + D\sigma_{ce}^2 \qquad \text{(B-29)}$$

$$\sigma_{X_j}^2 = D^2 \sigma_{ac'}^2 + D\sigma_{ae}^2 \qquad \text{(B-30)}$$

where σ_{ce}^2 and σ_{ae}^2 are the average variances

caused by random fluctuations which vary from segment to segment at the course.

The deviation is determined by taking the positive square roots of (B-29) and (B-30). Factoring out D, these become

$$\sigma_{X_J} = D\sqrt{\sigma^2_{ac'} + \frac{\sigma^2_{ac}}{D}} \qquad \text{(B-31)}$$

$$\sigma_{Y_J} = D\sqrt{\sigma^2_{cc'} + \frac{\sigma^2_{cc}}{D}} \qquad \text{(B-32)}$$

Dividing through by D to obtain the along-course and cross-course deviation per mile,

$$\text{along-course deviation} = \sqrt{\sigma^2_{ac'} + \frac{\sigma_{ac}}{D}} \qquad \text{(B-33)}$$

$$\text{cross-course deviation} = \sqrt{\sigma^2_{cc'} + \frac{\sigma_{cc}}{D}} \qquad \text{(B-34)}$$

Thus, it is shown that there should be two terms in the expressions for the along-course and cross-course deviations per mile. One of these terms is independent of the length of the trip and the other varies with the length of the trip as shown in (B-33) and (B-34).

A study has been made (9) of the various component

errors, and the expected mean value for these errors. Errors are found to be of three basic types:

1) Doppler ground speed (or along-course velocity) error, E_V, and Doppler drift error, E_D.
2) Heading reference error, E_H.
3) Computer error, E_C.

The total probable error is taken as the rms sum of these errors.

Most present-day Doppler navigation systems are of the Janus type (both forward and rearward looking with three or four beams). The velocity error for this type of system is (29)

$$E_V = \frac{dv}{v} = 1 - \cos \delta\gamma + \sin \delta\gamma \tan \alpha \qquad \text{(B-35)}$$

where

E_V = fractional velocity error

δv = error in velocity

V = velocity

$\delta\gamma$ = uncertainty in pitch angle

γ = angle between horizontal velocity component and direction of radiation

α = angle of climb or descent

For a typical γ of 70°, the uncertainty is only

0.02 per cent per degree. The choice of 70° as optimum value of γ represents a compromise between high velocity sensitivity (cps per knot) which increases with smaller γ angles, and high signal return over the sea which increases with larger γ angles.

The receiver of a Doppler navigation system must filter the spectrum of received frequencies to find the frequency associated with the most probable ground speed and drift. Figure B-3 shows a typical Doppler spectrum.

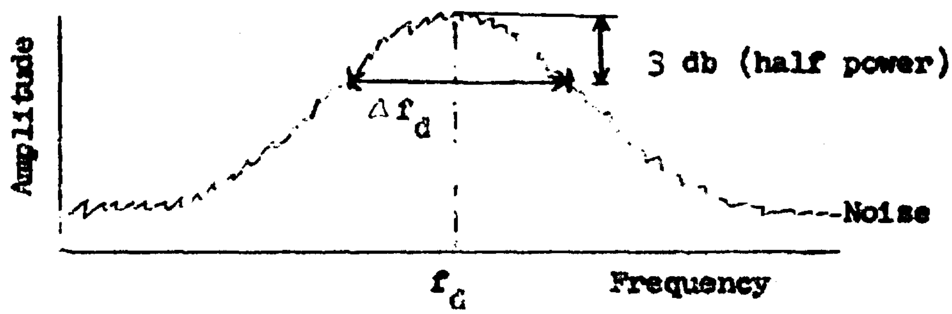

Figure B-3

Typical Doppler Spectrum

The smoothing time necessary to determine f_d is usually on the order of one second. This leads to large errors in the reading of instantaneous velocities. However, after longer periods, the fluctuation error becomes negligible. For example, after 100 seconds, the error is only 0.057%. The longer the

duration of the flight, the smaller this quantity becomes.

Since the electrical circuits cannot be made without noise and component inaccuracies, two forms of error are possible. The frequency tracker error is a function of both the Doppler frequency spectrum and the circuit used. This error is generally less than 0.1% of the velocity, but can increase at high altitude due to loss of signal strength. The second possible error is in the frequency of transmission. This is normally negligibly small with the use of automatic frequency control equipment.

When a Doppler radar is used over water, the velocity accuracy decreases somewhat; this can be attributed to three causes:

1) An increased terrain-bias error
2) An error due to surface droplet motion of the water
3) Water current motion

The most important of these is perhaps the first--an increased terrain-bias error. It is due to the marked change in scattering coefficient with looking angle over the extent of the radiated beamwidth. Figure B-4 (28) is a set of plots of the measured scattering coefficient vs. incidence angle (the complement of the γ angle for a β angle of zero) for an

X-band radar (8800-10,000 Mc) measured in the horizontal plane from the longitudinal axis of the aircraft to the projection of the radar beam. Figure B-4 shows the increased slope of the curves for the various sea state conditions as compared with the land curves near the commonly used 70° γ angle. This increased slope results in a slight skewing and substantial shift of the center of gravity of the received spectrum (Figure B-3), in contrast to that which would be obtained from the same velocity over land. The magnitude of the resulting bias error depends largely on the beamwidth of the antenna, decreasing rapidly with smaller beamwidths. Figure B-5 (28) is a plot of this bias error in percentage of velocity vs. beamwidth for a γ angle of 69° and a β angle of zero for two extremes of sea state, namely Beaufort 1 and Beaufort 4. Beaufort 1 is normally defined as "smooth sea, small wavelets (less than one foot)," and Beaufort 4 is normally defined as "rough sea, medium waves (five to eight feet)," although it is very difficult to determine these definitions accurately.

Since this error is a systematic or bias error, it is possible to eliminate it either by careful calculation or by inflight calibration for any one particular sea state. Most modern Doppler systems are equipped with

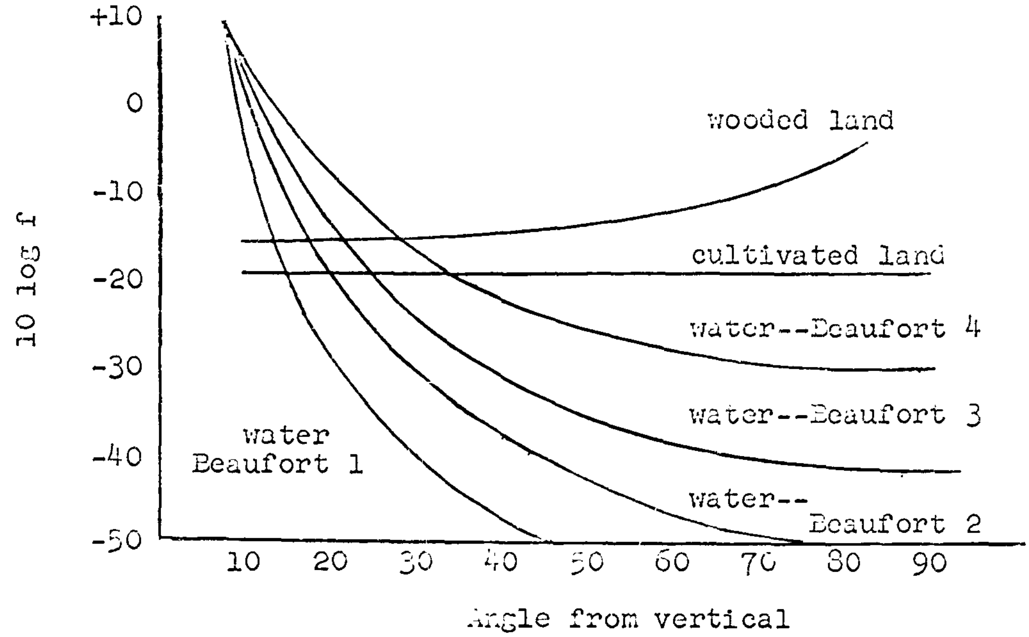

Figure B-4

Scattering Coefficients for Land and Water

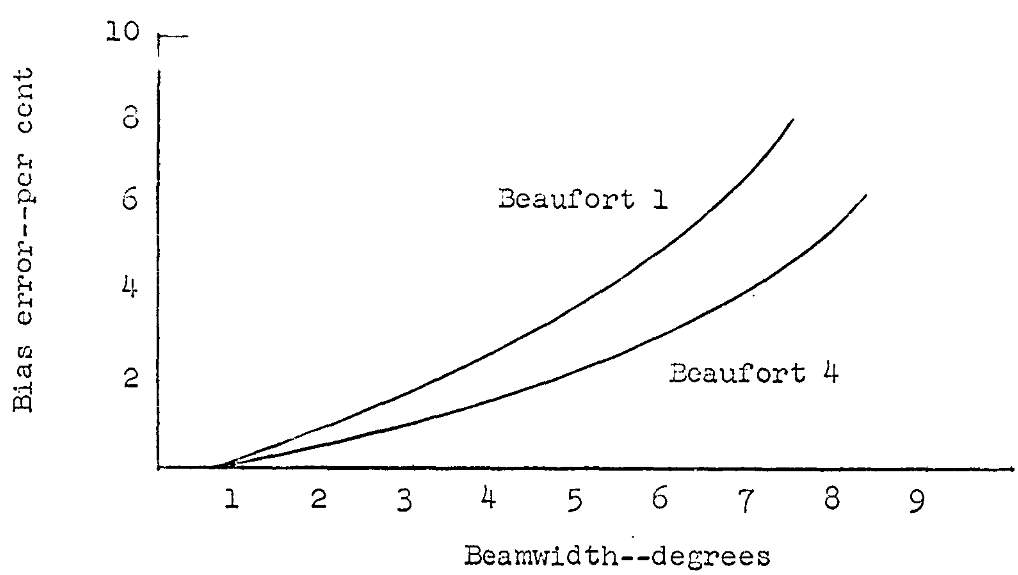

Figure B-5

Doppler Veclocity Water Bias Error vs. Beamwidth

a land-sea switch, with one sea position which can be calibrated for the most frequently occurring sea state. The error which remains is the spread between the errors for the sea state extremes. These errors are appreciably reduced when the β angle is other than zero.

The second error is caused by motion of the water surface droplets (and hence scatterers) due to the action of the wind. This has been determined experimentally to be somewhere between 8 and 16% of the wind speed, and can be disregarded, except in the case of extremely high winds.

The third over-water error is that due to current motion. This rate appears as a direct error in the ground speed measurement since the aircraft is measuring its velocity with respect to a moving reference. If this rate is known, it can be accounted for in post-flight analysis.

It might be mentioned that water "wave motion", as such, produces no Doppler error, since water mass is not actually transported or moved forward in the wave action.

Tests made by the Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio, conclude that velocity errors over water should be between 0.2 and 0.7% of the actual ground speed.

The second basic error in Doppler navigation

systems is the heading error E_H . This tends to swamp all other errors, and thus is the major contributing factor in accounting for the rather large circular probable error of Doppler systems. A one degree heading error gives a 1.8% error in present position. Since an inertial platform capable of giving an accurate heading reference will be on board the aircraft, the autopilot should be able to hold to within .25% of the desired heading. This will result in a 0.5% error in position. It should be noted that cross-track position is not nearly as essential as accurate ground speed for the determination of gravity. Ground speed enters the Eötvös correction directly, whereas heading enters only as a trigonometric function.

The third basic error is in the computer used for actual readout of ground speed and drift. Modern computers used in the Doppler system can count accuracies to within 0.2%.

The overall Doppler rms error may then be taken as

$$E_P = \sqrt{E_V^{\,2} + E_D^{\,2} + E_H^{\,2} + E_C^{\,2}} \qquad \text{(B-36)}$$

where E_P is the positional error in per cent.

A series of tests made (28) for 788 flight legs during the testing of the AN/APN-66 Doppler system showed errors which are empirically stated as

1) Per cent range error standard deviation $= 8/\sqrt{D}\%$ where D is the total distance traveled in nautical miles

2) Transverse error standard deviation $= 16 / \sqrt{D}\%$

3) Probable position error $= 11/\sqrt{D}$

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